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Trading off generations: Equity, discounting, and climate change ☆

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ABSTRACT

The prevailing literature discusses intergenerational trade-offs in climate change predominantly in terms of the Ramsey equation relying on the infinitely lived agent model. We discuss these trade-offs in a continuous time OLG framework and relate our results to the infinitely lived agent setting. We identify three shortcomings of the latter: first, underlying normative assumptions about social preferences cannot be deduced unambiguously. Second, the distribution among generations living at the same time cannot be captured. Third, the optimal solution may not be implementable in overlapping generations market economies.

1. Introduction

How much should society invest into avoiding or at least extenuating anthropogenic climate change? A key determinant of the optimal mitigation and investment levels is the social discount rate and a heated debate has evolved over its quantification. We analyze whether the infinitely lived agent (ILA) model employed in this debate is suitable to discuss the involved intergenerational trade-offs. For our analysis, we develop a new continuous time overlapping generations (OLG) growth model and compare the discounting formulas resulting from the ILA and the OLG framework. Our approach uncovers normative assumptions of calibration-based approaches to climate change assessment and explores equity and consistency concerns in normative approaches that refuse intergenerational discounting.

The Stern (2007) review on the economics of climate change, carried out by the former World Bank Chief Economist on behalf of the British government, has drawn significant attention in the political arena. It implies an optimal carbon tax that differs by an order of magnitude from the optimal tax derived by Nordhaus (2008) in his widely known integrated

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assessment model DICE.[1] Nordhaus (2007) shows that this difference is almost fully explained by the different assumptions on social discounting as summarized in the Ramsey equation.[2] Nordhaus himself favors a *positive* approach to social discounting using a calibration-based procedure that attempts to avoid explicit normative assumptions. In contrast, Stern (2007) advocates a *normative* approach emphasizing that only ethical considerations are valid to address the intergenerational trade-off.

The debate over the right discount rate almost exclusively relies on the Ramsey equation. The Ramsey equation characterizes how an ILA trades off consumption possibilities at different points in time. Contributors to the climate change discussion usually interpret the ILA framework as a utilitarian social welfare function, associating each point in time with the utility of a different generation. The real world is inhabited by overlapping generations, who value their own future consumption and possibly that of future generations. Barro (1974) shows that appropriate assumptions on altruism and operational bequests imply that finitely lived overlapping generations aggregate into a representative ILA. However, recent empirical studies indicate that the altruistic bequest motive is rather weak.[3] As a consequence, the dominant share of savings is driven by individual life-cycle planning rather then by altruistic transfers for future generations. Therefore, a calibration of the Ramsey equation to observed interest rates will necessarily reflect preference parameters that deal with individuals' life-cycle planning over their finite lifetime. Confined to an ILA framework, the current discounting debate is not capable of disentangling a social planner's discounting of future generations from an individual's discounting of his own future utility.

For our analysis, we develop a novel continuous time OLG model around two desiderata. First, in order to relate as closely as possible to the standard Ramsey equation, we choose a model in continuous time where agents live a finite deterministic life span. In contrast to the models based on Yaari (1965) and Blanchard (1985), where agents have an infinite lifetime and a constant probability of death, our model explicitly captures life-cycles. Second, we incorporate economic growth via exogenous technological change in order to make reasonable statements about intergenerational distribution. This feature is also a crucial distinction from the most closely related model in the literature by d'Albis (2007) who examines the influence of demographic structure on capital accumulation. Similar to Calvo and Obstfeld (1988), Burton (1993) and Marini and Scaramozzino (1995), we introduce a social planner maximizing the discounted life time utilities of the OLG.

Our analysis derives several theorems on the observational equivalence (identical macroeconomic aggregates) between the OLG frameworks and the ILA model. However, we show that the seemingly positive calibration of an ILA model to observed market outcomes involves normative assumptions. In particular, these assumptions imply that the social planner's pure rate of time preference is higher than that of the individuals living in the economy. Moreover, we show that the normative approach to discounting in the ILA setting overlooks a conflict between intergenerational equity and distributional equity among generations alive. Finally, we find that a social planner who is limited to tax labor and capital income cannot achieve the first-best social optimum without age-discriminatory tax schedules.

Related to our analysis, Aiyagari (1985) showed that under certain conditions an overlapping generations model with two-period-lived agents exhibits the same paths of aggregate capital and consumption as the discounted dynamic programming model with infinitely lived agents in discrete time. We complement these results by explicitly deriving the relation between the preference parameters of the OLG model and the observationally equivalent ILA framework in continuous time. The equivalence between the social planner solution in a continuous time OLG setting and an ILA model was already observed by Calvo and Obstfeld (1988). While they focus on time inconsistencies in fiscal policy, our focus is on intergenerational trade-offs.

Several environmental economic applications employ numerical simulations of integrated assessment models to compare interest rates and climate policy between ILA models and OLG frameworks in which agents live for two or three periods. Gerlagh and van der Zwaan (2000) point at differences between the models as a consequence of aging and distributional policies. Howarth (1998) compares the simulation results of a decentralized OLG, a constrained, and an unconstrained utilitarian OLG to the results obtained by Nordhaus (1994) using the ILA model DICE. While the decentralized OLG yields similar results as DICE, he finds substantial differences for the utilitarian OLGs. Calibrating time preference, Howarth (2000) shows that the unconstrained utilitarian OLG model and the ILA model can produce similar outcomes. Stephan et al. (1997) provide a simulation yielding equivalence between a decentralized OLG with bounded rationality and an ILA economy with limited foresight. In contrast, our model elaborates the analytical conditions under which the continuous time ILA and OLG frameworks are observationally equivalent. Burton (1993) and Marini and Scaramozzino (1995) analyze the relationship between individual welfare maximization and the optimal outcome of a benevolent social planner in an overlapping generations model with resources or environmental pollution. With this literature, our paper shares the insight that OLG models provide crucial insights about intergenerational trade-offs that

[1] Integrated assessment models augment economic growth models with a climate module, directly considering feedbacks between economic activity and climate change.

[2] In line with the environmental economic literature we call the Euler equation of the Ramsey–Cass–Koopmans growth model "Ramsey equation".

[3] See, e.g., Hurd (1987, 1989), Kopczuk and Lupton (2007), Laitner and Juster (1996), Laitner and Ohlsson (2001), Wilhelm (1996). These papers suggest either that the bequest motive is statistically insignificant, economically irrelevant, or, if there is a considerable bequest motive, that it is not of the altruistic type (in the sense of Barro, 1974 and Becker, 1974) but originates from other sources such as the "joy of giving". In all these cases an OLG economy does not reduce to an ILA economy.

cannot be captured in infinitely lived agent models. The next section explains the positive and normative approaches to social discounting and lays out the further structure of the paper.

2. Nordhaus, Stern and the relation between ILA and OLG models

The integrated assessment literature and the social discounting debate abstract from the real world OLG economy to an ILA model. Integrated assessment models either calibrate an ILA economy to the real world or fill in preference parameters based on ethical arguments. Then, the ILA is interpreted as a social planner evaluating climate policy. However, an OLG world reveals household preferences based on life cycle investment decisions. A social planner evaluating climate change faces a time horizon exceeding that of individual life cycle planning and his decisions affect future generations.

The majority of economists in the climate change debate takes an observation-based approach to social discounting. This view is exemplarily laid out in Nordhaus' (2007) critical review of the Stern (2007) review of climate change. Individual preferences towards climate change mitigation cannot be observed directly in market transactions because of the public good characteristic of greenhouse gas abatement. However, we observe everyday investment decisions on capital markets that carry information on intertemporal preferences. In particular, we observe the market interest rate and the steady state growth rate of the economy. The positive approach translates this information into (pairs of) time preference and a measure for the intertemporal elasticity of substitution. Then, this ILA is interpreted as a utilitarian social planner who confronts the climate problem in an integrated assessment model.

The normative approach to social discounting aims at treating all generations alike and, therefore, argues that a positive rate of time preference is non-ethical. This view is supported by a number of authors including Ramsey (1928), Pigou (1932), Harrod (1948), Koopmans (1965), Solow (1974), Broome and Schmalensee (1992) and Cline (1992). The Stern (2007) review of climate change effectively uses a zero rate of time preference, but adopts the parameter value $\rho^R = 0.1\%$ in order to capture a small but positive probability that society becomes extinct.[4]

Our major presumption is that the world looks more like an overlapping generations model than an infinitely lived agent framework. Accordingly, we interpret the real world (without policy intervention) as a decentralized OLG economy. In Section 3, we develop the decentralized, continuous time OLG model and establish conditions for existence and uniqueness of a steady state. Section 4 recalls the ILA Ramsey–Cass–Koopmans economy employed in the current discounting debate. Section 5 analyzes the relation between the preference parameters of OLG households and the ILA for observationally equivalent economies. In Section 6, we introduce a social planner into the OLG model and examine the relationship between this utilitarian OLG economy, the ILA model, and the decentralized OLG economy. We consider the case where the utilitarian social planner can fully control the economy as well as the situation where the planner is limited to non-age discriminatory taxes on labor and capital income. Section 7 discusses the consequences of our findings for intergenerational discounting and the debate on climate change mitigation. We show how the relations between the different models uncover normative assumptions in the seemingly positive ILA approach, and how the generational equity trade-off is more intricate than suggested by the normative ILA approach to climate change evaluation. Section 8 concludes.

3. An OLG growth model in continuous time

We introduce an OLG exogenous growth model in continuous time and analyze the long-run individual and aggregate dynamics of a decentralized economy in market equilibrium.

3.1. Households

Consider a continuum of households, each living the finite time span T. All households exhibit the same intertemporal preferences irrespective of their time of birth $s \in (-\infty,\infty)$. We assume that if households are altruistic, their altruistic preferences are not sufficiently strong for an operative bequest motive. This allows us to abstract from altruism in individual preferences. As a consequence, all households maximize their own welfare U, which is the discounted stream of instantaneous utility derived from consumption during their lifetime

$$U(s) \equiv \int_s^{s+T} \frac{c(t,s)^{1-(1/\sigma^H)}}{1-\frac{1}{\sigma^H}} \exp[-\rho^H(t-s)]\, dt, \tag{1}$$

where $c(t,s)$ is the consumption at calender time t of households born at time s, σ^H is the constant intertemporal elasticity of substitution and ρ^H denotes the constant rate of (pure) time preference of the households. Each household is endowed with one unit of labor at any time alive, which is supplied inelastically to the labor market at wage $w(t)$. In addition,

[4] Strictly speaking this is not time preference, but Yaari (1965) shows the equivalence of discounting because of a probability of death/extinction and a corresponding rate of time preference. Our superscript R labels inputs to the Ramsey equation.

households may save and borrow assets $b(t,s)$ at the interest rate $r(t)$. The household's budget constraint is[5]

$$\dot{b}(t,s) = r(t)b(t,s) + w(t) - c(t,s), \quad t \in [s,s+T]. \tag{2}$$

Households are born without assets and are not allowed to be indebted at the time of death. Thus, the following boundary conditions apply for all generations s

$$b(s,s) = 0, \quad b(s+T,s) \geq 0. \tag{3}$$

Because of the non-operative bequest motive, intertemporal welfare U of a household born at time s always increases in consumption at time $s+T$. Thus, in the household optimum the second boundary condition in Eq. (3) holds with equality.

Maximizing Eq. (1) for any given s subject to conditions (2) and (3) yields the well known Euler equation

$$\dot{c}(t,s) = \sigma^H [r(t) - \rho^H] c(t,s), \quad t \in [s,s+T]. \tag{4}$$

The behavior of a household born at time s is characterized by the system of differential equations (2) and (4) and the boundary conditions for the asset stock (3).

At any time $t \in (-\infty,\infty)$ the size of the population $N(t)$ increases at the constant rate $v \geq 0$. Normalizing the population at time $t=0$ to unity implies the birth rate γ[6]

$$N(t) \equiv \exp[vt] \;\Rightarrow\; \gamma = \frac{v \exp[vT]}{\exp[vT]-1}. \tag{5}$$

3.2. Firms

Consider a continuum of identical competitive firms $i \in [0,1]$. All firms produce a homogeneous consumption good under conditions of perfect competition from capital $k(t,i)$ and *effective labor* $A(t)l(t,i)$. $A(t)$ characterizes the technological level of the economy and grows exogenously at a constant rate ξ. Normalizing technological progress at $t=0$ to unity implies

$$A(t) \equiv \exp[\xi t]. \tag{6}$$

All firms have access to the same production technology $F(k(t,i),A(t)l(t,i))$, which exhibits constant returns to scale and positive but strictly decreasing marginal productivity with respect to both inputs capital and effective labor. Furthermore, F satisfies the Inada conditions.

Constant returns to scale of the production function and symmetry of the firms allow us to work with a representative firm whose decision variables are interpreted as aggregate variables. With minor abuse of notation, we introduce aggregate capital per effective labor, $k(t)$, and aggregate capital per capita, $\overline{k}(t)$,

$$k(t) \equiv \frac{\int_0^1 k(t,i)\,di}{A(t)\int_0^1 l(t,i)\,di}, \quad \overline{k}(t) \equiv \frac{\int_0^1 k(t,i)\,di}{N(t)}. \tag{7}$$

In addition, we define the intensive form production function $f(k(t)) \equiv F(k(t),1)$.

Profit maximization of the representative firm yields for the wage $w(t)$ and the interest rate $r(t)$

$$w(t) = A(t)[f(k(t)) - f'(k(t))k(t)], \tag{8a}$$

$$r(t) = f'(k(t)). \tag{8b}$$

3.3. Market equilibrium and aggregate dynamics

In order to investigate the aggregate dynamics of the economy, we introduce aggregate household variables per effective labor by integrating over all living individuals and dividing by the product of technological level and the labor force of the economy. Analogously to Eq. (7) we define under slight abuse of notation per effective labor household variables, $x(t)$, and aggregate household variables per capita, $\overline{x}(t)$,

$$x(t) \equiv \frac{\int_{t-T}^t x(t,s)\gamma \exp[vs]\,ds}{A(t)\int_0^1 l(t,i)\,di}, \quad \overline{x}(t) \equiv \frac{\int_{t-T}^t x(t,s)\gamma \exp[vs]\,ds}{N(t)}, \tag{9}$$

where $x(t,s)$ stands for the individual household variables consumption $c(t,s)$ and assets $b(t,s)$.

Assuming that all markets are in equilibrium at all times t implies the following aggregate dynamics of the economy[7]:

$$\frac{\dot{c}(t)}{c(t)} = \sigma^H [r(t) - \rho^H] - (v+\xi) - \frac{\Delta c(t)}{c(t)}, \tag{10a}$$

[5] Throughout the paper, partial derivatives are denoted by subscripts (e.g., $F_k(k,l) = \partial F(k,l)/\partial k$), derivatives with respect to calendar time t are denoted by dots and derivatives of functions depending on one variable only are denoted by primes.

[6] The equation is derived by solving $\int_{t-T}^t \gamma \exp[vs]\,ds = N(t)$, where $\gamma \exp[vs]$ denotes the cohort size of the generation born at time s. Observe that $\gamma \to 1/T$ for $v \to 0$ and $\gamma \to v$ for $T \to \infty$. Anticipating definition (13), we can also write $\gamma = 1/Q_T(v)$.

[7] Note that $\dot{x}(t) = -(v+\xi)x(t) + \exp[-(v+\xi)t]\int_{t-T}^t \dot{x}(t,s)\gamma \exp[vs]\,ds + \gamma[x(t,t) - x(t,t-T)/\exp[(v+\xi)T]]\exp[-\xi t]$.

$$\dot{k}(t) = f(k(t)) - (v+\xi)k(t) - c(t), \tag{10b}$$

where the term[8]

$$\Delta c(t) \equiv \frac{\gamma \exp[v(t-T)]c(t,t-T) - \gamma \exp[vt]c(t,t)}{\exp[vt]\exp[\xi t]}. \tag{10c}$$

captures the difference in aggregate consumption per effective labor between the generation born and the generation dying at time t. Substituting the individual household's Euler equation (4) into the aggregate Euler equation (10a) and recalling that $\dot{c}(t)/c(t) = \dot{\overline{c}}(t)/\overline{c}(t) - \xi$ according to (9), yields the following corollary:

Lemma 1 (*Sign of $\Delta c(t)/c(t)$*)
$\Delta c(t)/c(t) > 0$ *if and only if*

$$\frac{\dot{c}(t,s)}{c(t,s)} > \frac{\dot{\overline{c}}(t)}{\overline{c}(t)} + v \quad \textit{for all } s \in [t-T, t]. \tag{11}$$

As the right hand side of inequality (11) represents the growth rate of aggregate consumption, Lemma 1 states that $\Delta c(t)/c(t)$ is positive if and only if individual consumption grows faster than aggregate consumption.

3.4. Steady state

Our analysis will concentrate on the long-run steady state growth path of the economy, in which both consumption per effective labor and capital per effective labor are constant over time, i.e., $c(t) = c^\star$, $k(t) = k^\star$. From Eq. (8) follows that in the steady state the interest rate $r(t) = r^\star \equiv f'(k^\star)$ is constant and the wage $w(t)$ grows at the rate of technological progress ξ. The wage relative to the technology level is constant in the steady state

$$w^\star \equiv \left.\frac{w(t)}{\exp[\xi t]}\right|_{k=k^\star} = f(k^\star) - f'(k^\star)k^\star. \tag{12}$$

For $T \in \mathbb{R}_{++}$ we define the function $Q_T : \mathbb{R} \to \mathbb{R}_+$ as

$$Q_T(r) \equiv \frac{1-\exp[-rT]}{r}, \quad \forall\, r \neq 0, \tag{13}$$

and $Q_T(0) \equiv T$. $Q_T(r)$ can be interpreted as the present value of an annuity received over T years, at the discount rate r. Properties of the function Q_T are summarized in Lemma 3 in Appendix A.9. Expressing steady state consumption and wealth of individual households relative to the technology level returns functions that only depend on the household's *age* $a \equiv t-s$:

$$c^\star(a) \equiv \left.\frac{c(t,s)}{\exp[\xi t]}\right|_{k=k^\star} = w^\star \frac{Q_T(r^\star-\xi)}{Q_T(r^\star - \sigma^H(r^\star-\rho^H))} \exp[(\sigma^H(r^\star-\rho^H)-\xi)a], \tag{14a}$$

$$b^\star(a) \equiv \left.\frac{b(t,s)}{\exp[\xi t]}\right|_{k=k^\star} = w^\star Q_a(r^\star - \sigma^H(r^\star-\rho^H)) \exp[(r^\star-\xi)a] \times \left[\frac{Q_a(r^\star-\xi)}{Q_a(r^\star-\sigma^H(r^\star-\rho^H))} - \frac{Q_T(r^\star-\xi)}{Q_T(r^\star-\sigma^H(r^\star-\rho^H))}\right]. \tag{14b}$$

Fig. 1 illustrates these steady state paths for individual consumption and assets in terms of the technological level of the economy.[9] The individual consumption path grows exponentially over the lifetime of each generation. Individual household assets follow an inverted U-shape, i.e., households are born with no assets, accumulate assets in their youth and consume their wealth towards their death.

Applying the aggregation rule (9), we obtain for the aggregate values per effective labor

$$c^\star = w^\star \frac{Q_T(r^\star-\xi)}{Q_T(v)} \frac{Q_T(v+\xi-\sigma^H(r^\star-\rho^H))}{Q_T(r^\star-\sigma^H(r^\star-\rho^H))}, \tag{15a}$$

$$b^\star = \frac{w^\star}{r^\star-\xi}\left[\frac{Q_T(\xi+v-r^\star)}{Q_T(v)} - 1\right] - \frac{w^\star}{r^\star-\sigma^H(r^\star-\rho^H)} \times \frac{Q_T(r^\star-\xi)}{Q_T(v)} \frac{Q_T(\xi+v-r^\star) - Q_T(\xi+v-\sigma^H(r^\star-\rho^H))}{Q_T(r^\star-\sigma^H(r^\star-\rho^H))}. \tag{15b}$$

The following proposition guarantees the existence of a non-trivial steady state for a large class of production functions including Cobb–Douglas and CES production functions.

Proposition 1 (*Existence of the steady state*).
There exists a $k^\star > 0$ solving Eqs. (8) *and* (15) *with $b^\star = k^\star$ if*

$$\lim_{k\to 0} -kf''(k) > 0. \tag{16}$$

The proof is given in the Appendix.

[8] Note that $\Delta c(t)$ includes via $c(t,t-T)$ and $c(t,t)$ all values of $k(s)$ for $s \in [t-T, t+T]$. Thus, (10) defines a system of integro-differential equations. In the steady state, however, $\Delta c(t)/c(t) = \sigma^H[r^\star - \rho^H] - (v+\xi)$, where $r^\star$ denotes the steady state interest rate.

[9] The calculations use the following model specifications: $f(k) = k^\alpha$, $\alpha = 0.3$, $\rho = 3\%$, $\sigma = 1$, $\xi = 1.5\%$, $v = 0$, $T = 50$.

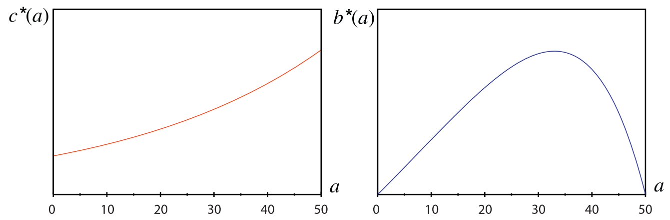

Fig. 1. Steady state paths of consumption (left) and asset (right) for individual households over age.

Intuitively, due to the Inada conditions, we obtain $f(k)>k$ for sufficiently small k and $f(k)<k$ for k sufficiently large. These conditions imply a fixed point $b^\star=k^\star$ if the savings rate $1-c(t)/f(k)$ is sufficiently large for $k\to 0$, which is guaranteed by condition (16). In the proof of Proposition 1 we show that steady states may be equal to or larger than the golden rule capital stock k^{gr}, which is implicitly defined by $r^{gr}\equiv \nu+\xi=f'(k^{gr})$. As our aim is to compare the decentralized OLG with an ILA economy, we are particularly interested in steady states with $k^\star<k^{gr}$.[10]

Definition 1 (*Decentralized OLG economy*).

(i) The set $\Gamma\equiv\{f,\xi,\nu,\sigma^H,\rho^H,T\}$ defines a decentralized OLG economy.
(ii) $\Gamma^\star\in\{\Gamma\,|\,\exists\, k^\star$ with $0<k^\star<k^{gr}\}$ defines a decentralized OLG economy with a dynamically efficient capital stock $k^\star<k^{gr}$. For an economy $\Gamma^\star$ we refer by $k^\star$ and $r^\star$ to a steady state satisfying this condition.

The following proposition shows the existence of dynamically efficient economies $\Gamma^\star$. Analogously to d'Albis (2007), we introduce the share of capital in output, $s(k)$, and the elasticity of substitution between capital and labor, $\epsilon(k)$,

$$s(k)\equiv\frac{kf'(k)}{f(k)},\quad \epsilon(k)\equiv-\frac{f(k)-f'(k)k}{k^2f''(k)}. \tag{17}$$

Proposition 2 (*Existence and uniqueness of dynamically efficient steady states*).
Given that condition (16) holds, there exists a steady state with $k^\star<k^{gr}$ if

$$\frac{k^{gr}}{f(k^{gr})-r^{gr}k^{gr}}>\frac{Q_T{}'(\nu)}{Q_T(\nu)}-\frac{Q_T{}'((\nu+\xi)(1-\sigma^H)+\sigma^H\rho^H)}{Q_T((\nu+\xi)(1-\sigma^H)+\sigma^H\rho^H)}. \tag{18}$$

There exists exactly one $k^\star<k^{gr}$ if

$$s(k)\le\epsilon(k)\quad\textit{and}\quad \frac{d}{dk}\left(\frac{s(k)}{\epsilon(k)}\right)\ge 0, \tag{19a}$$

and, in case that $\sigma^H>1$,

$$\rho^H<\frac{\sigma^H-1}{\sigma^H}(\nu+\xi). \tag{19b}$$

The proof is given in the Appendix.

Consider a small increase in k. It implies a decrease in the interest rate r and an increase in wage w, their relative increase being reflected in the elasticity $\epsilon(k)$. On the one hand, an increase in k positively affects the households' incomes, which would ceteris paribus lead to higher savings. On the other hand, a decrease in r also affects the households' saving rates. The conditions in (19a) ensure that the increase in the aggregate saving rate is sufficiently small (potentially negative) that the marginal increase in b by a marginal increase in k remains below one, which guarantees a unique steady state $k^\star=b^\star$. Although we cannot solve the implicit equation $k^\star=b^\star$ analytically and, therefore, cannot calculate the steady state interest rate $r^\star$, the following proposition determines a lower bound of the steady state interest rates in a dynamically efficient OLG economy.

Lemma 2 (*Lower bound of steady state interest rate*).
For any economy Γ^* (which implies $r^\star>\nu+\xi$) holds

$$r^\star>\rho^H+\frac{\xi}{\sigma^H}. \tag{20}$$

The proof is given in the Appendix.

A steady state interest rate $r^\star$ satisfying condition (20) ensures that per capita consumption increases at a higher rate than aggregate consumption, which has to hold for aggregate savings to be positive.

[10] In the ILA economy steady states with $k^\star\ge k^{gr}$ are ruled out by the transversality condition (23).

4. Infinitely lived agent economy and observational equivalence

As intergenerational trade-offs are mostly discussed in ILA frameworks rather than in OLG models, we investigate how the macroeconomic observables of an OLG and ILA economy relate to each other. Therefore, we first introduce the ILA model and then define observational equivalence between two economies. Whenever we compare two different model structures in this paper we assume that population growth and the production side of the economy are identical.

Variables of the ILA model that are not exogenously fixed to its corresponding counterparts in the OLG model are indexed by a superscript ^{R}. The ILA model abstracts from individual generations' life cycles only considering *aggregate* consumption and asset holdings. In the ILA model, optimal consumption and asset paths per capita derive from maximizing the discounted stream of instantaneous utility of consumption per capita weighted by population size

$$U^R \equiv \int_0^\infty N(t) \frac{\overline{c}(t)^{1-(1/\sigma^R)}}{1-(1/\sigma^R)} \exp[-\rho^R t]\, dt, \tag{21}$$

subject to the budget constraint

$$\dot{\overline{b}}(t) = [r(t)-v]\overline{b}(t) + w(t) - \overline{c}(t). \tag{22}$$

and the transversality condition

$$\lim_{t\to\infty} \overline{b}(t) \exp\left[-\int_0^t r(t')\, dt' + vt\right] = 0. \tag{23}$$

In the following we assume that the transversality condition is met.[11] The solution to the ILA's maximization problem is characterized by (22), (23) and the well-known Ramsey equation

$$r(t) = \rho^R + \frac{\overline{g}(t)}{\sigma^R}, \tag{24}$$

where we denote per capita consumption growth as $\overline{g}(t) = \dot{\overline{c}}(t)/\overline{c}(t)$. In the steady state we have $\overline{g}(t) = \xi$. If markets are in equilibrium at all times (i.e., $\int_0^1 l(t,i)\, di = N(t)$ and $k(t) = b(t)$), the system dynamics of the ILA model in terms of effective labor is given by:

$$\frac{\dot{c}(t)}{c(t)} = \sigma^R[r^R(t) - \rho^R] - \xi, \tag{25a}$$

$$\dot{k}(t) = f(k(t)) - (v+\xi)k(t) - c(t). \tag{25b}$$

To compare the different models we use the following definition:

Definition 2 (*Observational equivalence*).

(i) Two economies A and B are *observationally equivalent* if coincidence in their current observable macroeconomic variables leads to coincidence of their future observable macroeconomic variables. Formally, if for any $c^A(0) = c^B(0)$ and $k^A(0) = k^B(0)$ it holds that $c^A(t) = c^B(t)$ and $k^A(t) = k^B(t)$ for all $t \geq 0$.
(ii) Two economies A and B are *observationally equivalent in steady state* if there exist $c^\star$ and $k^\star$ such that both economies are in a steady state.

Note that observational equivalence in the steady state (ii) is weaker than general observational equivalence (i).

5. Decentralized OLG versus infinitely lived agent economy

Now, we investigate under what conditions a decentralized OLG economy, as outlined in Section 3, is observationally equivalent to an ILA economy, as defined in Section 4. The following proposition states the necessary and sufficient condition:

Proposition 3 (*Decentralized OLG versus ILA economy*).

(i) *A decentralized OLG economy $\Gamma^\star$ and an ILA economy are observationally equivalent if and only if for all $t \geq 0$ the following condition holds*:

$$\rho^R = \frac{\sigma^H}{\sigma^R}\rho^H + \left(1 - \frac{\sigma^H}{\sigma^R}\right) r(t) + \frac{1}{\sigma^R}\left[\frac{\Delta c(t)}{c(t)} + v\right]. \tag{26}$$

[11] In the steady state, the transversality condition holds if $\rho^R > (1-1/\sigma^R)\xi + v$.

(ii) *For any decentralized OLG economy* $\Gamma^{\star}$ *there exists an ILA economy that is observationally equivalent in the steady state.*
(iii) *If a decentralized OLG economy* $\Gamma^{\star}$ *is observationally equivalent in the steady state to an ILA economy, the following statements hold*:
(a) *For* $\sigma^R = \sigma^H$:

$$\rho^R = \rho^H + \frac{1}{\sigma^R}\left[\frac{\Delta c(t)}{c(t)} + v\right] > \rho^H. \tag{27}$$

(b) *In general*:

$$\rho^R > \rho^H \Leftrightarrow \sigma^R > \sigma^H\left[1 + \frac{1}{\xi}\left(\frac{\Delta c(t)}{c(t)} + v\right)\right]^{-1}. \tag{28}$$

The proof is given in the Appendix.

Proposition 3 states that any decentralized OLG economy $\Gamma^{\star}$ is – at least in the steady state – observationally equivalent to an ILA economy for an appropriate choice of (σ^R, ρ^R). Note that (σ^R, ρ^R) is, in general, not uniquely determined by (26).

If we assume that the intertemporal propensity to smooth consumption between two periods is the same for the households in the OLG and the ILA economy, i.e., $\sigma^H = \sigma^R$, we obtain that $\rho^R > \rho^H$ in the steady state. To understand why the pure rate of time preference in the ILA economy exceeds the corresponding rate in the observationally equivalent OLG economy, we analyze the term $[\Delta c(t)/c(t) + v]$, which is strictly positive in the steady state.

The first part, $\Delta c(t)/c(t)$, captures the difference in consumption between the cohort dying and the cohort just born relative to aggregate consumption. The term is a consequence of the fact that every individual in the OLG model plans his own life cycle, saving while young and spending while old. We know from Lemma 1 that $\Delta c(t)/c(t) > 0$ if and only if individual consumption grows faster than aggregate consumption, which is always satisfied if there is no population growth, i.e., $v = 0$.

The second part, v, reflects that instantaneous utility in the ILA model is weighted by population size. Hence, for a growing population future consumption receives an increasing weight in the objective function. A corresponding weighting does not occur in the decentralized OLG economy, where all households only maximize own lifetime utility. As a consequence, the time preference rate of an observationally equivalent ILA must be higher to compensate for the greater weights on future consumption.

Equipping an ILA with a lower intertemporal substitutability than the household in the decentralized OLG economy would ceteris paribus increase the steady state interest rate in the ILA economy (as opposed to the situation with coinciding elasticities). In order to match the same observed interest rate as before, the ILA's rate of time preference has to be lower. Thus, the time preference relation can flip around if picking the intertemporal elasticity of substitution of the ILA sufficiently below that of the household in the decentralized OLG economy.

6. Utilitarian OLG versus infinitely lived agent economy

Consider an OLG economy, which is governed by a social planner maximizing a social welfare function. In this section, we investigate the conditions under which this economy is observationally equivalent to an ILA economy. We assume a utilitarian social welfare function in which the social planner trades off the weighted lifetime utility of different generations. The weight consists of two components. First, the lifetime utility of the generation born at time s is weighted by cohort size. Second, the social planner exhibits a social rate of time preference $\rho^S > 0$ at which he discounts the expected lifetime utility at birth for generations born in the future.[12]

Assuming that the social planner maximizes social welfare from $t=0$ onward, the social welfare function consists of two parts: (i) the weighted integral of the *remaining* lifetime utility of all generations alive at time $t=0$, and (ii) the weighted integral of all future generations

$$W \equiv \int_{-T}^{0}\left\{\int_{0}^{s+T}\frac{c(t,s)^{1-(1/\sigma^H)}}{1-(1/\sigma^H)}\exp[-\rho^H(t-s)]\,dt\right\}\gamma\exp[vs]\exp[-\rho^S s]\,ds$$
$$+\int_{0}^{\infty}\left\{\int_{s}^{s+T}\frac{c(t,s)^{1-(1/\sigma^H)}}{1-(1/\sigma^H)}\exp[-\rho^H(t-s)]\,dt\right\}\gamma\exp[vs]\exp[-\rho^S s]\,ds. \tag{29a}$$

The term in the first curly braces is the (remaining) lifetime utility $U(s)$ of a household born at time s, as given by Eq. (1), the functional form of which is a given primitive for the social planner. The term $\gamma\exp[vs]$ denotes the cohort size of the

[12] We examine the discounted utilitarian social welfare function of, e.g., Calvo and Obstfeld (1988), Burton (1993) and Marini and Scaramozzino (1995), as it represents the de facto standard in the economic literature. For a general criticism of discounted utilitarianism, as also employed in the climate change debate by Nordhaus (2007) and Stern (2007), see, e.g., Sen and Williams (1982) and Asheim and Mitra (2010). Calvo and Obstfeld (1988) show that social welfare functions which do not treat all present and future generations symmetrically, i.e., discount lifetime utility to the same point of reference (here the date of birth), may lead to time-inconsistent optimal plans.

generation born at time s. Changing the order of integration and replacing $t-s$ by age a, we obtain

$$W=\int_0^\infty \left\{ \int_0^T \frac{c(t,t-a)^{1-(1/\sigma^H)}}{1-(1/\sigma^H)} \gamma \exp[(\rho^S-\rho^H-\nu)a]\, da \right\} \exp[(\nu-\rho^S)t]\, dt. \tag{29b}$$

In the following, we consider two different scenarios. In the *unconstrained* utilitarian OLG economy, a social planner maximizes the social welfare function (29b) directly controlling investment and household consumption. Thus, the social planner is in command of a centralized economy. In contrast, in the *constrained* utilitarian OLG economy the social planner relies on a market economy, in which the households optimally control their savings and consumption maximizing their individual lifetime utility (1). In this second scenario, the social planner is constrained to influencing prices by a tax/subsidy regime in order to maximize the social welfare function (29b).

6.1. Unconstrained utilitarian OLG economy

We determine the unconstrained social planner's optimal allocation by maximizing (29b) subject to the budget constraint (10b) and the transversality condition

$$\lim_{t\to\infty} k(t)\exp\left[-\int_0^t f'(k(t'))\, dt' + (\xi+\nu)t\right] = 0. \tag{30}$$

Following the approach of Calvo and Obstfeld (1988), we interpret the unconstrained social planner's optimization problem as two nested optimization problems. The first problem is obtained by defining

$$V(\overline{c}(t)) \equiv \max_{\{c(t,t-a)\}_{a=0}^T} \int_0^T \frac{c(t,t-a)^{1-(1/\sigma^H)}}{1-(1/\sigma^H)} \gamma \exp[(\rho^S-\rho^H-\nu)a]\, da, \tag{31}$$

subject to

$$\int_0^T c(t,t-a)\gamma \exp[-\nu a]\, da \le \overline{c}(t). \tag{32}$$

The solution to this maximization problem is the social planner's optimal distribution of consumption between all generations alive at time t.

Proposition 4 (*Optimal consumption distribution for given time t*).
The optimal solution of the maximization problem (31) subject to condition (32) is

$$c(t,t-a) = \overline{c}(t) \frac{Q_T(\nu)}{Q_T(\nu+\sigma^H(\rho^H-\rho^S))} \exp[-\sigma^H(\rho^H-\rho^S)a]. \tag{33}$$

As a consequence, all households receive the same amount of consumption at time t irrespective of age for $\rho^H=\rho^S$, and receive less consumption the older (younger) they are at a given time t for $\rho^H>\rho^S$ ($\rho^H<\rho^S$).

The proof is given in the Appendix.

Proposition 4 states that the difference between the households' rate of time preference ρ^H and the social rate of time preference ρ^S determines the social planner's optimal distribution of consumption across households of different age at some given time t. In particular, if $\rho^H>\rho^S$ the consumption profile with respect to age is qualitatively opposite to that of the decentralized solution at any time t, as following from the Euler equation (4) and illustrated in Fig. 2.[13] That is, in the social planner's solution households receive less consumption the older they are, whereas they would consume more the older they are in the decentralized OLG economy. The intuition for this result is as follows. The social planner weighs the lifetime utility of every individual discounted to the time of birth. Thus, the instantaneous utility at time t of those who are younger (born later) is discounted for a relatively longer time at the social planner's time preference (before birth) and for a relatively shorter time by the individual's time preference (after birth) than is the case for the instantaneous utility at time t of those who are older (born earlier). For $\rho^H>\rho^S$ the social planner's time preference is smaller and, thus, the young generation's utility at time t receives higher weight.

Proposition 4 shows that the standard approach of weighted intergenerational utilitarianism poses a trade-off between *intertemporal* generational equity and *intratemporal* generational equity to the social planner whenever households exhibit a positive rate of time preference. Lifetime utilities of today's and future generations would receive equal weight if and only if the social rate of time preference were zero. Approaching this by a close to zero social time preference rate, $\rho^H>\rho^S\approx 0$ implies that at each point in time the young enjoy higher consumption than the old.[14] In contrast, an equal distribution of consumption among the generations alive is obtained if and only if social time preference matches

[13] We do not take up a stance on the relationship between the individual and the social rate of time preference, but merely hint at the resulting consequences. This is in line with Burton (1993) and Marini and Scaramozzino (1995), who argue that they represent profoundly different concepts and, thus, may differ. In fact, ρ^H trades off consumption today versus consumption tomorrow *within* each generation, while ρ^S trades off lifetime utilities *across* generations. If they are supposed to differ, then it is usually assumed that $\rho^H>\rho^S$ (see also Heinzel and Winkler, 2011 and von Below, 2012).

[14] Note that for $\rho^S=0$ the maximization problem of the unconstrained social planner is not well defined.

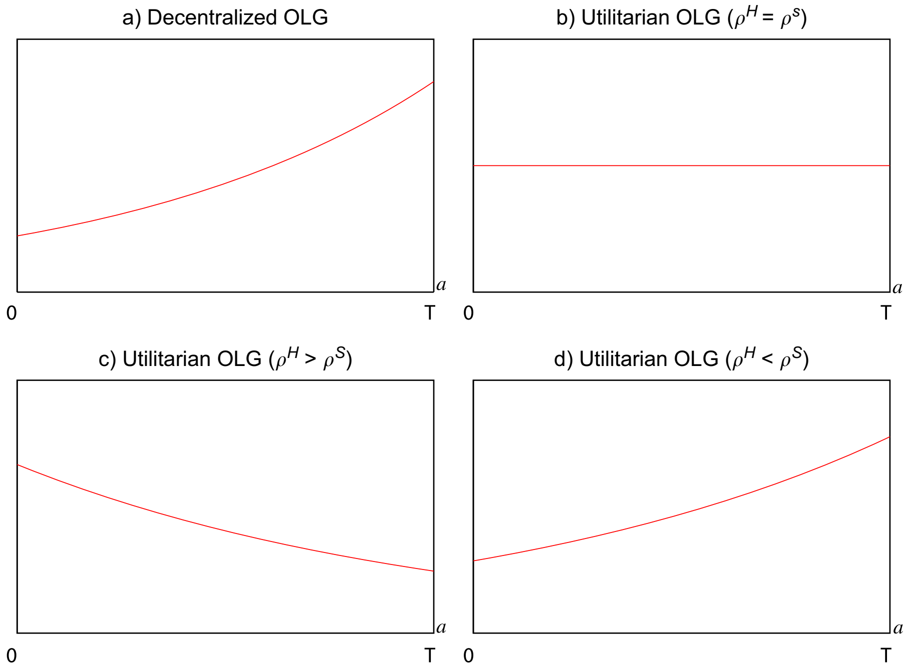

Fig. 2. Distribution of consumption across all generations alive at given time t dependent on age a for the decentralized OLG and three different utilitarian OLGs.

individual time preference. However, a positive social rate of time preference comes at the expense of an unequal treatment of lifetime utilities of different generations. This trade-off practically vanishes only if the individuals' and the social planner's rates of time preference are both very close to zero. Such an equality trade-off can only be captured in an OLG model which explicitly considers the life cycles of different generations.

We now turn to the second part of the maximization problem, which optimizes $\overline{c}(t)$ over time. It is obtained by replacing the term in curly brackets in Eq. (29b) by the left hand side of Eq. (31) resulting in

$$\max_{\{\overline{c}(t)\}_{t=0}^{\infty}} \int_0^{\infty} V(\overline{c}(t))\exp[\nu t]\exp[-\rho^S t]\, dt, \tag{34}$$

subject to the budget constraint (10b). Observe that problem (34) is formally equivalent to an ILA economy with the instantaneous utility function $V(\overline{c}(t))$ and the time preference rate ρ^S.[15] We obtain $V(\overline{c}(t))$ by inserting the optimal consumption profile (33) into Eq. (31) and carrying out the integration

$$V(\overline{c}(t)) = \left[\frac{Q_T(\nu+\sigma^H(\rho^H-\rho^S))}{Q_T(\nu)}\right]^{1/\sigma^H} \frac{\overline{c}(t)^{1-(1/\sigma^H)}}{1-(1/\sigma^H)}. \tag{35}$$

The social planner's maximization problem (34) is invariant under affine transformations of the objective function (35), in particular, under a multiplication with the inverse of the term in square brackets. Thus, problem (34) is identical to the optimization problem in the ILA economy when setting the intertemporal elasticity of substitution $\sigma^R = \sigma^H$ and the time preference rate $\rho^R = \rho^S$.

Proposition 5 (*Unconstrained utilitarian OLG and ILA economy*).
For an unconstrained utilitarian OLG economy, i.e., a social planner maximizing the social welfare function (29b) subject to the budget constraint (10b) and the transversality condition (30), the following statements hold:

(i) *An unconstrained utilitarian OLG economy is observationally equivalent to the ILA economy if and only if* $\sigma^R = \sigma^H$ *and* $\rho^R = \rho^S$.
(ii) *An unconstrained utilitarian OLG economy is observationally equivalent in the steady state to an ILA economy if and only if*

$$\rho^R = \rho^S + \xi \frac{\sigma^R - \sigma^H}{\sigma^R \sigma^H}. \tag{36}$$

The proof is given in the Appendix.

[15] Such an equivalence was already observed by Calvo and Obstfeld (1988).

Proposition 5 states that maximizing the utilitarian social welfare function (29b) yields the same aggregate consumption and capital paths as maximizing the welfare (21) in the ILA model with $\sigma^R=\sigma^H$ and $\rho^R=\rho^S$. This result, however, does not imply that the unconstrained social planner problem can, in general, be replaced by an ILA model.

First, to derive the equivalence result, we have assumed a social planner who does *not* exhibit any preferences for smoothing lifetime utility across generations. The parameter σ^H in Eq. (35) stems from the individuals' preferences to smooth consumption within the lifetime of each generation. It is therefore a given primitive to the social planner. Thus, the only normative parameter the social planner may choose is the social time preference rate ρ^S. It remains an open question for future research whether a different welfare functional for the unconstrained utilitarian social planner exists that permits a normative choice of σ^S for the social planner and still delivers observational equivalence to an ILA model with $\rho^S=\rho^R$.

Second, in the ILA setting, the first-best solution is easily decentralized, e.g., using taxes that ensure the optimal path of the aggregate capital stock. However, such implementation may fail in the case of the unconstrained social planner, because he is also concerned about the intratemporal allocation of consumption across all generations alive at a certain point in time. Before we investigate the decentralization of the social optimum in the next section, we compare the outcome of the OLG economy managed by the unconstrained social planner to that of a decentralized OLG economy. In all comparisons between a utilitarian and a decentralized OLG economy, we assume identical preferences of the individual households in both economies.

Proposition 6 (*Unconstrained utilitarian OLG and decentralized OLG*).

(i) *For any economy Γ^* there exists an unconstrained utilitarian OLG that is observationally equivalent in the steady state. In such a steady state $\rho^S>\rho^H$.*
(ii) *In the steady state, an economy Γ^* and an unconstrained utilitarian OLG exhibit the same allocation of consumption across the generations alive at each point in time if and only if they are observationally equivalent in the steady state.*

The proof is given in the Appendix.

Remark 1. The converse of (i) is not true, as there exists no economy Γ^* that would be observationally equivalent to an unconstrained utilitarian OLG with $\rho^S<\rho^H$.

Proposition 6 implies that an unconstrained utilitarian OLG economy exhibits the same aggregate steady state as the decentralized OLG economy if and only if the intratemporal distribution of consumption between all generations alive coincide. For this to hold, the social planner's rate of time preference has to be higher than the individual households' rate of time preference.

6.2. Constrained utilitarian OLG economy

As seen in Proposition 6, the optimal solution of a social planner maximizing (29b) subject to the budget constraint (10b) and the transversality condition (30) is, in general, not identical to the outcome of a decentralized OLG economy.[16] Thus, the question arises whether and if so how the social optimum is implementable in a decentralized market economy. Calvo and Obstfeld (1988) show that it is possible to implement the social optimum by a transfer scheme discriminating by date of birth s and age a. Such a transfer scheme may be difficult to implement because of its administrative burden. In addition, it is questionable whether taxes and subsidies which are conditioned on age per se are politically viable.[17]

As a consequence, we consider a social planner that cannot discriminate transfers by age but may only influence prices via taxes and subsidies. In particular, we assume that the social planner may impose taxes/subsidies on capital and labor income. Let $\tau_r(t)$ and $\tau_w(t)$ denote the tax/subsidy on returns on savings and on labor income, respectively.[18] The individual households of the OLG economy base their optimal consumption and saving decisions on the effective interest rate $r^e(t,\tau_r(t))$ and the effective wage $w^e(t,\tau_w(t))$ defined by

$$r^e(t,\tau_r(t))=r(t)-\tau_r(t), \tag{37a}$$

$$w^e(t,\tau_w(t))=w(t)[1-\tau_w(t)]. \tag{37b}$$

[16] Recall that we assume the individual preference parameters to be identical in both economies.

[17] We observe policies that redistribute wealth between generations living at the same time, e.g., in education, health care and old-age pensions. However, we argue that these redistributions use age as a proxy for health condition, or particular needs, and redistribute from high to low income levels rather than redistributing because of age per se. Note that the "Age Discrimination Act of 1975" for the US states explicitly that "…no person in the United States shall, on the basis of age, be excluded from participation in, be denied the benefits of, or be subjected to discrimination under, any program or activity receiving Federal financial assistance." In particular, we consider it unlikely that taxes such as value added tax, income tax, capital tax, which are rather conditioned on income could instead be conditioned on age. As a consequence, we consider the possibility of age discriminating taxation and redistribution as rather limited.

[18] Following the standard convention, $\tau_i(t)$ is positive if it is a tax and negative if it is a subsidy.

Then, the individual budget constraint reads

$$\dot{b}^{e}(t,s) = r^{e}(t,\tau_r(t))\, b^{e}(t,s) + w^{e}(t,\tau_w(t)) - c^{e}(t,s). \tag{37c}$$

Given this budget constraint, individual households choose consumption paths which maximize lifetime utility (1). Thus, the optimal consumption path $c^{e}(t,s,\{r(t'),\tau_r(t'),\tau_w(t')\}_{t'=s}^{s+T})$ is a function of the paths of the interest rate $r(t)$ and the taxes $\tau_r(t)$ and $\tau_w(t)$.

Note that for a given path of the interest rate and given tax/subsidy schemes $\{r(t),\tau_r(t),\ \tau_w(t)\}_{t=s}^{s+T}$ the individual household's optimal paths of consumption and assets can be characterized as in the decentralized OLG economy by (2) and (4) when using $r^{e}(t,\tau_r(t))$ and $w^{e}(t,\tau_w(t))$ instead of $r(t)$ and $w(t)$, respectively. Applying the aggregation rule (9) yields aggregate consumption per effective labor $c^{e}(t,\{r(t'),\tau_r(t'),\tau_w(t')\}_{t'=t-T}^{t+T})$. To analyze observational equivalence between such a constrained utilitarian OLG economy and an ILA economy, we have to restrict redistribution to mechanisms which do not alter the aggregate budget constraint (10b) of the economy. We consider the following redistribution scheme which yields a balanced government budget at all times

$$\tau_w(t)w(t) = -\tau_r(t)\overline{b}(t). \tag{37d}$$

Under these conditions the social optimum is, in general, not implementable.

Proposition 7 (*Implementation of the social optimum*).
The optimal solution of a social planner maximizing (29b) subject to the budget constraint (10b) and the transversality condition (30) is not implementable by a tax/subsidy regime satisfying (37) unless this solution is identical to the outcome of the unregulated decentralized OLG economy $\Gamma^{\star}$.

The proof is given in the Appendix.

Proposition 7 states that a constrained social planner who can only impose a tax/subsidy regime on interest and wages cannot achieve the first-best social optimum. The intuition is that the constrained social planner can achieve the socially optimal aggregate levels of capital and consumption, but cannot implement the socially optimal intratemporal distribution of consumption across generations living at the same time. The only exception occurs if the social optimum happens to be identical to the outcome of the unregulated OLG economy. In this case, there is no need for the social planner to interfere and, thus, it does not matter whether the social planner can freely re-distribute consumption among generations or is constrained to a self-financing tax/subsidy scheme. In all other cases, the constrained social planner will choose a tax path such as to achieve a second-best optimum. In consequence, Proposition 7 questions the validity of the ILA model in deriving distributional policy advice for a democratic government that may be limited in conditioning redistribution between generations on age.

7. Stern versus Nordhaus – a critical review of choosing the social rate of time preference

A prime example for questions of intergenerational distribution is the mitigation of anthropogenic climate change, as most of its costs accrue today while the benefits spread over decades or even centuries. The question of optimal greenhouse gas abatement has been analyzed in integrated assessment models combining an ILA economy with a climate model. Interpreting the ILA's utility function (21) as a utilitarian social welfare function, intergenerational equity concerns are closely related to the choice of intertemporal elasticity of substitution σ^R and the rate of time preference ρ^R. This is illustrated well by Nordhaus (2007), who compares two runs of his open source integrated assessment model DICE-2007. The first run uses his preferred specifications $\sigma^R = 0.5$ and $\rho^R = 1.5\%$. The second run employs $\sigma^R = 1$ and $\rho^R = 0.1\%$, which are the parameter values chosen by Stern (2007). These different parameterizations cause a difference in the optimal reduction rate of emissions in the period 2010–2019 of 14% versus 53% and a difference in the optimal carbon tax of 35$ versus 360$ per ton C.

The previous sections derived important differences between the OLG economy and an ILA model, which have immediate implications for the evaluation of climate change mitigation polices. This section relates our findings to the positive and to the normative approach to social discounting.

7.1. The "positive" approach

Under strong assumption on altruism, Barro (1974) interprets finitely lived overlapping generations as a dynasty and shows how to represent them as an ILA. If we are interested in dynastic welfare as a whole, then a set of dynasties alive today will efficiently distribute resources across time under the assumption that all investments are private and there exist complete and undistorted future markets. In this context, a project evaluation that uses a discount rate different from the market interest would generally be inefficient.[19] However, the Pareto-efficiency argument only holds for the different dynasties as a whole. If we are concerned with the welfare of individual generations, any project evaluation rate will make

[19] This statement only holds for small projects. A large intertemporal transfer can change intertemporal prices and, thus, the current market rate is no longer the efficient interest and discount rate when implementing the project.

some future generations better-off, at the expense of others. Thus, the rate at which we evaluate projects characterizes a particular distribution of intergenerational welfare, none of which Pareto dominates any other.[20]

In our framework, where we explicitly account for the different generations and assume that there is no operative bequest motive,[21] this feature becomes even more salient: individual households in the decentralized OLG economy live for a finite time span T during which they exclusively save for their own consumption in old age. As a consequence, market observations and, in particular, the market interest rate do not reveal any information on households' preferences concerning the intergenerational distribution of welfare. The standard positive approach proceeds in two steps. First, it calibrates an ILA to match real world observation, in particular the real interest rate. Second, it interprets the ILA as a utilitarian social planner who evaluates a public project. Applying the first step of the positive approach, we showed in Proposition 3 that the rate of time preference of the ILA does not reflect the actual time preference of the (homogeneous) individuals in the decentralized OLG economy. In particular, if we set the ILA's elasticity of intertemporal substitution equal to that of the households ($\sigma^R = \sigma^H$), then the ILA model overestimates the rate of pure time preference for two reasons ($\rho^R > \rho^H$). First, the ILA plans for an infinite future when taking his market decisions. Households in the OLG economy, however, only plan for their own lifespan when revealing their preferences on the market. Interpreting these decisions as if being taken with an infinite time horizon overstates their pure time preference. Second, the ILA model assumes that the representative consumer accounts for population growth by giving more weight to the welfare of the larger future population, a concern absent in the welfare maximization of the households in the OLG economy.

In the second step, the positive approach interprets the ILA framework as a social planner economy. Proposition 5 indeed verifies observational equivalence between the ILA framework and an OLG economy with an unconstrained utilitarian social planner economy in an OLG world. However, Proposition 6 reveals that the pure rate of time preference ρ^S that is implicitly assumed for the social planner is larger than the pure rate of time preference of the individuals living in the economy: $\rho^S > \rho^H$.[22] In particular, the finding holds for the feasible interpretation $\rho^R = \rho^S$ and $\sigma^R = \sigma^H$, which associates pure time preference of the ILA with that of the social planner.[23]

What are the normative assumptions of the positive approach in a world where overlapping generations only plan for their own life-cycles? The most important assumption is that the positive approach selects a particular intergenerational weight for the social planner that exceeds the pure rate of time preference of the individuals living in the economy. While we do not take a stance on the "right" relationship between the two rates, the positive approach has to justify its particular choice. As we pointed out in the beginning, the intergenerational weight does not derive from efficiency arguments, cannot be deduced from the real market interest rate, and has immediate distributional implications. Moreover, the assumption stands in sharp contrast to most of the literature on social discounting, which argues for an intergenerational discount rate that is equal to or lower than the households' pure time preference rate.

Let us illustrate how the standard positive approach implicitly manipulates discount rates and time horizons. We start by spelling out the positive welfare function underlying our OLG world, where households are fully selfish and do not care for future generations. It simply consists of the sum of the remaining lifetime utilities of the individuals presently alive (and is given by the first term in the welfare function 29a). Greenhouse gas mitigation would not be optimal in such an approach, if benefits of mitigation accrue beyond the lifetime of these individuals. The standard positive approach deviates in two accounts: First, it assumes an infinite time horizon. Second, it assigns more weight to larger future generations. In order to be consistent with market observations, both of these deviations force the standard approach to increase the rate of pure time preference. Thus, in our OLG world, the standard positive approach seems contradictory: first, it assigns higher weights to future generations. Then, it crowds these weights out again by increasing impatience.

If the world is correctly represented by overlapping generations that only care for their own lifetime utility, why should we be concerned, from a purely positive perspective, about long-run problems such as climate change? Just because intergenerational preferences are not represented in market transactions reflecting life-cycle savings does not necessarily imply that households do not exhibit such preferences. Many potential frictions of the socio-economic environment may lead to an incomplete expression of household preferences (e.g., missing or incomplete markets, public good properties, imperfect political representation). Then, a positive approach has to elicit intergenerational preferences in a non-market environment. Votes on long-term public investments might be a promising setting to elicit such preferences.[24] The remaining assumption in such an approach will have to deal with the precise mapping of, e.g., voting outcomes to the social planner's intergenerational discount rate.

[20] Efficiency dictates that we undertake more lucrative investments first. As Nordhaus (2007) emphasizes, if an investment in man-made capital is more efficient in raising future welfare than an investment into natural capital, we first have to invest in man-made capital. Our framework does not address the optimal investment portfolio, but focuses on intergenerational distribution.

[21] We interpret birth as appearance on the labor market. Hence, no operative bequest does not imply an absence of educational investment in children.

[22] The calibration in the first step relies on observational equivalence between the ILA and the decentralized economy, and the social planner interpretation in the second step relies on observational equivalence between the ILA and the utilitarian planner model. By transitivity of observational equivalence, we can therefore invoke Proposition 6.

[23] Alternative interpretations imply that σ^R does not match the intertemporal elasticity of the households σ^H. In particular, invoking observational equivalence while setting or interpreting $\rho^R = \rho^H$ would require an increase in the ILA's consumption discount rate by lowering the intertemporal elasticity of substitution to $\sigma^R < \sigma^H$.

[24] However, one has to account for problems related to intransitivities of voting outcomes, as discussed in Jackson and Yariv (2011).

If we cannot obtain reliable data on individual's intergenerational discount rates, we can complement positively observed individual preferences by transparent normative assumptions. The following approach seems particularly appealing: we adopt the normative assumption that the social planner should discount consumption at any given point in time independently of the birth date of the consuming generation. As shown in Proposition 4, this assumption implies that the intergenerational time preference of the social planner has to equal the pure rate of time preference of the individuals living in the economy. We can deduce the individuals' preference parameters from micro-estimates. However, depending on the context and method, estimates of the pure rate of time preference and the intertemporal elasticity of substitution vary by an order of magnitude.[25] Therefore, we suggest following the standard positive approach in validating the preference parameters based on their macro-economic implications (see below). We acknowledge that the drafted alternative approach contains normative assumptions. However, they are explicit and likely to be more reasonable in a world without an operative bequest motive than the implicit assumptions of the standard positive approach.

We close with a numeric illustration that shows how individual preferences deduced from the macroeconomic equilibrium differ from the observationally equivalent ILA preferences. First, assume an interest rate of $r=5.5\%$ and elasticities $\sigma^H=\sigma^R=.5$ as in Nordhaus (2008) latest version of DICE. Then the rate of pure time preference of the ILA is $\rho^R=1.5\%$, while the individuals of the decentralized OLG economy exhibit a time preference $\rho^H=-5.3\%$.[26] The surprising finding of a negative rate of time preference questions the plausibility of the above specifications. A simple sensitivity check suggests that increasing the intertemporal elasticity of substitution is most promising for resolving the negativity puzzle. The more recent asset pricing literature suggests an estimate of the intertemporal elasticity of substitution of $\sigma^H=1.5$ which, in combination with a disentangled measure of risk attitude, explains various asset prizing puzzles.[27] Adopting this estimate, we find $\rho^H=1.9\%$ for the households in the decentralized OLG economy and a time preference rate of $\rho^R=4.2\%$ for the ILA. The wide-spread assumption of logarithmic utility ($\sigma^H=1$) chosen by Stern (2007) implies that households have precisely the rate of pure time preference $\rho^H=0.1\%$ that the review chose for the social planner based on normative reasoning.

7.2. *The normative approach*

In a normative approach to social discounting it seems more natural to jump straight to an ILA model. By normatively justified assumptions the social planner exhibits an infinite planning horizon and particular values of the time preference rate and the intertemporal elasticity of substitution. It is obvious, however, that the ILA model cannot capture any distinction or interaction between intergenerational weighting and individual time preference. Nevertheless, Proposition 5 shows that a social planner fully controlling an OLG economy is observationally equivalent to an ILA economy if the parameters σ^R and ρ^R are appropriately chosen. In particular, the intertemporal path of aggregate consumption does not depend on the individual rate of time preference ρ^H, but only on the social planner's rate of time preference ρ^S. In fact, the time preference rate of the social planner coincides with the rate of time preference ρ^R of the observationally equivalent ILA economy. This finding provides some support for Stern's (2007) normative approach to intergenerational equity in the ILA model.

However, the shortcut of setting up an ILA economy exhibits a number of caveats as questions of intergenerational equity are more complex than the ILA model reveals. First, according to Proposition 5, the interpretation of the time preference rate of the ILA economy as the time preference rate of a social planner in an observationally equivalent social planner OLG economy ($\rho^R=\rho^S$) requires that the intertemporal elasticity of substitution in the ILA economy be equal to that of the individual households in the OLG economy, i.e., $\sigma^R=\sigma^H$. This constraint, however, implies that the intertemporal elasticity of substitution is a primitive to the social planner and cannot be chosen to match particular normative considerations.[28]

Second, interpreting the ILA economy as a utilitarian social planner OLG neglects the *intratemporal* allocation of consumption across all generations alive at each point in time. The utilitarian OLG model allows us to explicitly analyze the social planner's optimal intratemporal distribution of consumption. As shown in Proposition 4, it depends on the difference between the social planner's and the individual households' rates of time preference. Usually, it is assumed that the normatively chosen social rate of time preference ρ^S is smaller than the individual rate of time preference ρ^H.[29] According to Proposition 4, in this case the oldest generation receives least consumption while the newborns get most among all generations alive (see Fig. 2, part c). In contrast, the decentralized OLG economy would distribute relatively more to the old (see Fig. 2, part a). As a consequence, the standard discounted utilitarianism implies a trade-off between

[25] Considering household data, estimates for the pure rate of time preference range from around zero (Epstein and Zin, 1991; Browning et al., 1999) to about 10% (Andersen et al., 2008). Experimental studies find time preference rates exceeding even 20%, in particular, if not elicited jointly with the elasticity of intertemporal substitution (Harrison et al., 2005). Estimates for the elasticity of intertemporal substitution range from close to zero (Hall, 1988) to values around 2 (Chen et al., 2011).

[26] The calculation solves Eq. (15b) or, alternatively, $F(5.5\%)=J(5.5\%)$ in the notation introduced in the proof of Proposition 1. We choose the following exogenous parameters: capital share $\alpha=.3$, rate of technological progress $\xi=2\%$, rate of population growth $\nu=0\%$, and lifetime $T=50$.

[27] Vissing-Jørgensen and Attanasio (2003) estimate and Bansal and Yaron (2004) and Bansal et al. (2010) calibrate intertemporal substitutability to this value based on approaches employing Epstein and Zin (1991) preferences and Campbell (1996) log-linearization of the Euler equation.

[28] Note that the social welfare function (29b) we considered does not include any preferences for smoothing lifetime utility of different generations over time. Of course, such functional forms are conceivable but it is not clear whether and how such a utilitarian OLG economy translates into an observationally equivalent ILA economy.

[29] This assumption seems particularly reasonable if ρ^S is close to zero. With respect to the Stern review, it implies that the individual households' time preference rates exceed $\rho^S=0.1\%$.

intertemporal and *intratemporal* generational equity whenever households exhibit a positive rate of pure time preference. The aim of 'treating all generations alike' is therefore neither implemented easily in the economy nor captured in the utilitarian objective function.

Finally, there is an additional caveat, which applies to both the positive and the normative approach to social discounting. The ILA shortcut to the social planner OLG economy conceals that the first-best solution has to be implemented in a decentralized OLG instead of a Ramsey–Cass–Koopmans economy. In general, the social optimum not only requires re-distribution across time but also across different generations living at the same time. Apart, from the question whether consumption discrimination by age is justified on ethical grounds, it is questionable whether it is implementable (see footnote 17). In Proposition 7 we show that, in general, a social planner whose policy instruments are limited to non-age-discriminating taxes and subsidies cannot implement the first-best solution. In fact, the first-best social optimum can only be achieved in the special case that it coincides with the outcome of the decentralized OLG economy without any regulatory intervention. Thus, the ILA economy, interpreted as an unconstrained social planner model, cannot capture this second-best aspect of optimal policies.

8. Conclusions

In the climate change debate intergenerational trade-offs are most often discussed within ILA frameworks, which are interpreted as a utilitarian social welfare function. In this paper, we analyzed to what extent these models can represent the relevant intertemporal trade-offs if an altruistic bequest motive is non-operative.

We showed under which conditions an ILA economy is observationally equivalent to (i) a decentralized OLG economy and (ii) an OLG economy in which a social planner maximizes a utilitarian welfare function. We found that preference parameters differ in the decentralized OLG and the observationally equivalent ILA economy. In general, pure time preference of an ILA planner is higher than pure time preference of the households in the observationally equivalent OLG economy. Moreover, in a normative setting, a utilitarian social planner faces a trade-off between intergenerational and intragenerational equity that cannot be captured in the ILA model. Finally, the limited implementability of the first best allocation can only be observed and discussed in the OLG context.

Our results have important implications for the recent debate on climate change mitigation and, more generally, for ILA based integrated assessment and cost benefit analysis that relies on the Ramsey equation. First, the positive approach to specify the social welfare function implicitly assumes that the time preference rate of the social planner exceeds the one of the individual households. Second, the ILA model does not capture the distribution of consumption among generations alive at a given point in time. The utilitarian OLG model implies that a more equal treatment of lifetime utilities between present and future generations can come at the expense of a more unequal treatment of the generations alive at a given point in time—at least if individuals possess a positive rate of pure time preference. Thus, the utilitarian ILA in the normative approach to social discounting misses an important generational inequality trade-off. Third, the ILA approach overlooks a limitation in the implementability that arises if the intergenerational discount rate of the social planner in a utilitarian OLG economy does not coincide with the time preference rate of individual households. Then, the social optimum involves re-distribution among generations at each point in time, which would have to rely on age-discriminating taxes.

Our analysis employs two central assumptions. First, we assume selfish individual households. Although several empirical studies suggest that altruistic bequest motives are rather weak, extending the model to include different degrees of altruism is an interesting venue for future research. Second, part of our analysis assumes a specific utilitarian social welfare function. Although commonplace in the literature, this assumption drives some of our results, such as the trade-off between intra- and intergenerational equity. In particular, discounted utilitarianism in general has been questioned as an appropriate approach to deal with questions of intergenerational equity (e.g., Asheim and Mitra, 2010).

Appendix A

A.1. Proof of Proposition 1

To prove the existence of a non-trivial steady state, i.e. $k^\star \neq 0$, we follow closely part (A) of the proof of Proposition 2 in Gan and Lau (2010). We re-write Eq. (15b) for $r^* \notin \{\xi, \nu+\xi\}$ as[30]

$$b^\star = \frac{w^\star}{r^\star - \nu - \xi}\left\{\frac{Q_T(r^\star-\xi)}{Q_T(\nu)}\frac{Q_T(\nu+\xi-\sigma^H(r^\star-\rho^H))}{Q_T(r^\star-\sigma^H(r^\star-\rho^H))}-1\right\}. \tag{A.1}$$

We define the function $J:\mathbb{R}\to\mathbb{R}$ by

$$J(r) \equiv \frac{Q_T(r-\xi)}{Q_T(\nu)}\frac{Q_T(\nu+\xi-\sigma^H(r-\rho^H))}{Q_T(r-\sigma^H(r-\rho^H))}, \quad \forall\, r \in \mathbb{R}, \tag{A.2}$$

[30] The equivalence of Eq. (15b) and (A.1) is easily verified by multiplying over the terms in the denominator and expanding the resulting expressions. In addition, the domain of the functions making up the right hand side of Eqs. (15b) and (A.1) can be extended to $r^* \in \{\xi, \nu+\xi\}$ by limit. Both right hand side functions are continuous and coincide for these points. Thus, the two equations are equivalent for all $r^\star$.

for which Lemma 4 in Appendix A.9 summarizes some useful properties. Defining further

$$\phi(k) \equiv \frac{f(k)-f'(k)k}{f'(k)-v-\xi}[J(f'(k))-1], \tag{A.3}$$

the steady state is given by the solution of the equation $k=\phi(k)$, or equivalently

$$\lambda(k) \equiv \frac{J(f'(k))-1}{f'(k)-v-\xi} - \frac{k}{f(k)-f'(k)k} = 0. \tag{A.4}$$

Note that $\lambda(k)$ exhibits a removable pole at the golden rule capital stock k^{gr} which is given by $f'(k^{gr}) = v+\xi \equiv r^{gr}$. By defining

$$\lambda(k^{gr}) \equiv \lim_{k\to k^{gr}} \lambda(k) = J'(f'(k^{gr})) - \frac{k^{gr}}{f(k^{gr})-f'(k^{gr})k^{gr}}, \tag{A.5}$$

where we use l'Hospital's rule (recognizing that $J(f'(k^{gr}))=1$), we establish that $\lambda(k)$ is a well-defined and continuous function on $k \in \mathbb{R}$. We now show that

$$\lim_{k\to 0}\lambda(k) = +\infty, \quad \text{and} \quad \lim_{k\to\infty}\lambda(k) = -\infty, \tag{A.6}$$

which proves the existence of $k^\star \in (0,\infty)$ with $\lambda(k^\star)=0$ or equivalently $\phi(k^\star)=k^\star$.

For $k\to 0$, $f'(k)$ tends to ∞, $f(k)-f'(k)k$ tends to 0 and $J(f'(k))$ tends to ∞. The latter holds, as $\lim_{r\to\infty} J'(r)/J(r) > 0$ (see part (iii) and (v) of Lemma 4), which implies that $\lim_{r\to\infty} J(r) = +\infty$ and $\lim_{r\to\infty} J'(r) = +\infty$. Applying l'Hospital's rule we obtain

$$\lim_{k\to 0}\lambda(k) = \lim_{k\to 0} J'(f(k)) - \frac{1}{f''(k)k} = +\infty, \tag{A.7}$$

as $\lim_{k\to 0} 1/(f''(k)k)$ is finite by virtue of assumption (16).

For $k\to\infty$, $f(k)$ tends to ∞ and $f'(k)$ tends to 0. Thus, the first summand of $\lambda(k)$ tends to $[1-J(0)]/(v+\xi)$, which is finite. For the second summand observe that

$$\lim_{k\to\infty} \frac{f(k)-f'(k)k}{k} = \lim_{k\to\infty}\left[\frac{f(k)}{k} - f'(k)\right] = 0. \tag{A.8}$$

As $f(k)-f'(k)k > 0$ for $k>0$ this implies that $\lim_{k\to\infty} k/[f(k)-f'(k)k] = +\infty$ and, therefore, $\lim_{k\to\infty}\lambda(k) = -\infty$.

A.2. Proof of Proposition 2

To prove the proposition, we re-write the steady state condition (A.3) for $k \neq k^{gr}$ as

$$\frac{f(k)-(v+\xi)k}{f(k)-f'(k)k} = J(f'(k)), \tag{A.9}$$

which allows to distinguish between efficient and inefficient steady states. Moreover, we discuss solutions to Eq. (A.9) in terms of the interest rate r instead of the capital stock k. Therefore, we define

$$F(r) \equiv \frac{f(k(r))-(v+\xi)k(r)}{f(k(r))-f'(k(r))k(r)}, \tag{A.10}$$

where $k(r) = f'^{-1}(r)$, which is well defined due to the strict monotonicity of $f'(k)$. Observe that $k'(r) = 1/f''(k(r))$. The derivative of F with respect to r yields:

$$F'(r) = \frac{f'(k(r))-(v+\xi)}{f''(k(r))[f(k(r))-f'(k(r))k(r)]} + \frac{k(r)[f(k(r))-(v+\xi)k(r)]}{[f(k(r))-f'(k(r))k(r)]^2}. \tag{A.11}$$

Then, for $r^\star \neq r^{gr}$, a steady state is given by the solution of the equation $F(r^\star) = J(r^\star)$.

From (A.5) we observe that

$$J'(r^{gr}) = \frac{k^{gr}}{f(k^{gr})-r^{gr}k^{gr}} = F'(r^{gr}), \tag{A.12}$$

has to hold for $r=r^{gr}$ respectively $k=k^{gr}$ to be a steady state. In addition, we find for $r=r^{gr}$ that

$$F(r^{gr}) = 1 = J(r^{gr}). \tag{A.13}$$

From the proof of Proposition 1 follows that, given condition (16) holds, there exists an efficient steady state with $r^\star > r^{gr}$ and $k^\star < k^{gr}$ for $F'(r^{gr}) > J'(r^{gr})$. This can be seen from Eq. (A.5), which implies $\lambda(k^{gr}) < 0$, and $\lim_{k\to 0}\lambda(k) = \lim_{r\to\infty}\lambda(k(r)) = +\infty$. The condition $F'(r^{gr}) > J'(r^{gr})$ is equivalent to condition (18).

We now derive sufficient conditions such that there exists only one steady state $k^\star < k^{gr}$. Suppose that condition (16) holds, which guarantees existence of a dynamically efficient steady state. There exists only one steady state interest rate $r^\star$

with $r^\star > r^{gr}$ if and only if

$$F'(r)\big|_{r=r^\star} < J'(r)\big|_{r=r^\star} \quad \forall\, r^\star > r^{gr}$$
$$\Leftrightarrow \frac{F'(r)}{F(r)}\bigg|_{r=r^\star} < \frac{J'(r)}{J(r)}\bigg|_{r=r^\star} \quad \forall\, r^\star > r^{gr}. \tag{A.14}$$

The second line holds, as $F(r)=J(r)$ for all $r=r^\star$. A sufficient condition for (A.14) to hold is that

$$\frac{d}{dr}\left(\frac{F'(r)}{F(r)}\bigg|_{r=r^\star}\right)<0 \wedge \frac{d}{dr}\left(\frac{J'(r)}{J(r)}\bigg|_{r=r^\star}\right)>0 \quad \forall\, r^\star > r^{gr}. \tag{A.15}$$

From part (ii) and (iv) of Lemma 4 we know that the second condition holds for all $r > r^{gr}$ if, in case that $\sigma > 1$, also condition (19b) holds.

$$\frac{F'(r)}{F(r)}\bigg|_{r=r^\star} = \left[\frac{r-\nu-\xi}{f''(k(r))[f(k(r))-(\nu+\xi)k(r)]}+\frac{k(r)}{f(k(r))-rk(r)}\right]\bigg|_{r=r^\star} \tag{A.16a}$$

$$= \left[\frac{1}{k(r)f''(k(r))}\left(1-\frac{1}{F(r)}\right)+\frac{k(r)}{f(k(r))-rk(r)}\right]\bigg|_{r=r^\star} \tag{A.16b}$$

$$= \left[\frac{1}{k(r)f''(k(r))}\left(1-\frac{1}{J(r)}\right)+\frac{k(r)}{f(k(r))-rk(r)}\right]\bigg|_{r=r^\star} \tag{A.16c}$$

$$= \underbrace{\frac{k(r)}{f(k(r))-rk(r)}}_{\equiv g_1(r)}\left[1-\left(1-\frac{1}{J(r)}\right)\underbrace{\frac{f(k(r))-rk(r)}{-k^2(r)f''(k(r))}}_{\equiv g_2(r)}\right]\bigg|_{r=r^\star}. \tag{A.16d}$$

From the second to the third line we employed $F(r)=J(r)$ for all $r=r^\star$. We show in the following that $g_1'(r)\le 0$ and $g_2'(r)\ge 0$ are sufficient for $d/dr\left(F'(r)/F(r)|_{r=r^\star}\right)<0$.

First, observe from Eq. (A.3) that $J(r^\star)>1$ for all $r^\star > r^{gr}$. As $J(r)$ is U-shaped on $r\in(r^{rg},\infty)$ because of part (ii) and (iv) of Lemma 4 and $J(r^{gr})=1$, this implies that $J'(r^\star)>0$ for all $r^\star > r^{gr}$.

Second, we show that $F'(r)/F(r)|_{r=r^\star}>0$ for all $r^\star > r^{gr}$ if $g_2'(r)\ge 0$. Observe that

$$\lim_{r^\star\to\infty}\frac{F'(r)}{F(r)}\bigg|_{r=r^\star} = \lim_{r\to\infty}\left[\frac{1}{k(r)f''(k(r))}\left(1-\frac{1}{J(r)}\right)+\frac{k(r)}{f(k(r))-rk(r)}\right] \tag{A.17a}$$

$$= \lim_{r\to\infty}\left[\frac{1}{k(r)f''(k(r))}+\frac{k(r)}{f(k(r))-rk(r)}\right] \tag{A.17b}$$

$$= \lim_{r\to\infty}\left[\frac{1}{k(r)f''(k(r))}-\frac{1}{k(r)f''(k(r))}\right]=0. \tag{A.17c}$$

In addition, we know that $g_1(r)>0$ for all $r>0$ and

$$\lim_{r\to\infty} g_1(r) = \lim_{r\to\infty}\frac{1}{k(r)f''(k(r))}>0. \tag{A.18}$$

The latter implies together with Eq. (A.17)

$$\lim_{r\to\infty} g_2(r)\left(1-\frac{1}{J(r)}\right)=1. \tag{A.19}$$

As $g_2(r)(1-(1/J(r)))$ equals zero at $r=r^{gr}$ and is monotonically increasing in r for $g_2'(r)\ge 0=0$, this implies that $F'(r)/F(r)|_{r=r^\star}>0$ for all $r^\star > r^{gr}$. Then, we obtain for $g_1'(r)\le 0$ and $g_2'(r)\ge 0$

$$\frac{d}{dr}\left(\frac{F'(r)}{F(r)}\bigg|_{r=r^\star}\right)=g_1'(r)\left[1-\left(1-\frac{1}{J(r)}\right)g_2(r)\right]-g_1(r)g_2(r)\frac{J'(r)}{J^2(r)}-g_1(r)g_2'(r)\left(1-\frac{1}{J(r)}\right)<0. \tag{A.20}$$

The conditions $s(k)\ge\epsilon(k)$ and $d/dk(s(k)/\epsilon(k))$ are sufficient for $g_1'(r)\le 0$ and $g_2'(r)\ge 0$.

A.3. *Proof of Lemma 2*

We show that $\sigma(r^\star-\rho^H)-\xi>0$ is a necessary condition for aggregate assets $b^\star$ to be strictly positive in a dynamically efficient steady state, i.e., $(\sigma^H,\rho^H)\in\Gamma_{\Psi,T}$. As $b^\star=k^\star$ holds, this implies that for $k^\star>0$ the steady state real interest rate must exceed $\rho^H+\xi/\sigma$.

The household's wealth, as given by Eq. (14b), can be re-written to yield

$$b^\star(a) = \frac{w^\star}{r^\star - \xi}\{\theta \exp[(\sigma(r^\star-\rho^H)-\xi)a]+(1-\theta)\exp[(r^\star-\xi)a]-1\}, \tag{A.21}$$

with

$$\theta = \frac{1-\exp[-(r^\star-\xi)T]}{1-\exp[-(r^\star-\sigma^H(r^\star-\rho))T]}. \tag{A.22}$$

Assuming a dynamically efficient steady states implies that $r^\star-\xi>0$ and we obtain from (A.22)

$$\theta \begin{cases} <1 & \text{if } \sigma(r^\star-\rho^H)-\xi<0 \\ =1 & \text{if } \sigma(r^\star-\rho^H)-\xi=0 \\ >1 & \text{if } \sigma(r^\star-\rho^H)-\xi>0 \end{cases}. \tag{A.23}$$

Thus, we can directly infer from (A.21) that $b^\star(a)=0$ for all $a \in [0,T]$ for $\sigma(r^\star-\rho^H)-\xi=0$. As all households hold no assets, the aggregate capital stock equals zero. To show that $\sigma(r^\star-\rho^H)-\xi<0$ precludes strictly positive capital stocks, we analyze the second derivative of $b^\star(a)$

$$\frac{d^2\, b^\star(a)}{d\, a^2} = \frac{w^\star}{r^\star-\xi}\{\theta(\sigma(r^\star-\rho^H)-\xi)^2 \exp[(\sigma(r^\star-\rho^H)-\xi)a]+(1-\theta)(r^\star-\xi)^2\exp[(r^\star-\xi)a]\}. \tag{A.24}$$

For $\sigma(r^\star-\rho^H)-\xi<0$, $\theta<1$ holds, which implies that $d^2b^\star(a)/da^2>0$. Hence, the household's wealth profile is strictly convex. Together with the boundary conditions $b^\star(0)=0=b^\star(T)$ this implies that all households possess non-positive wealth at all times. This, in turn, precludes $k^\star>0$.

Further, it is obvious from (A.21) and (A.24) that $\sigma(r^\star-\rho^H)-\xi>0$ does not contradict strictly positive wealth of the individual households and, therefore, is a necessary condition for $k^\star>0$.

A.4. Proof of Proposition 3

(i) Both economies exhibit the same technology and rate of population growth by assumption and, thus, the market equilibria on the capital and the labor market imply that the equations of motion for the aggregate capital per effective labor (25b) and (10b) coincide. The remaining difference in the macroeconomic system dynamics is governed by the Euler equations (10a) and (25a) and by the transversality condition (23).

"⇒": Suppose the two economies are observationally equivalent, i.e., coincidence in the initial levels of consumption and capital imply coincidence at all future times. For this to hold the Euler equations (10a) and (25a) have to coincide giving rise to (26).

"⇐": If condition (26) holds, then also the Euler equations (10a) and (25a) coincide and the system dynamics of both economies is governed by the same system of two ordinary first order differential equations. The solution is uniquely determined by some initial conditions on c and k. Thus, if the two economies coincide in the levels of consumption and capital at one point in time they also do so for all future times. In consequence, the two economies are observationally equivalent. Moreover, the capital stock is an equilibrium of $\Gamma^\star$ implying $k^\star<k^{gr}$. As a consequence, the transversality condition for the ILA economy is satisfied and, thus, the described path is indeed an optimal solution.

(ii) Let $r^\star$ be the steady state interest rate of $\Gamma^\star$. Thus, all combinations of (ρ^R,σ^R) which satisfy

$$r^\star = \rho^R + \frac{\xi}{\sigma^R}, \tag{A.25}$$

yield ILA economies which are observationally equivalent in the steady state. As for all $\Gamma^\star$, $r^\star<r^{gr}$ holds, also the transversality condition (23) is satisfied.

(iii) The equality part of Eq. (27) follows directly from (26) by setting $\sigma^R=\sigma^H$. For the steady state, Eq. (10a) returns $1/\sigma^H[\Delta c(t)/c(t)+v]=r(t)-\rho^H-\xi/\sigma^H$ which, by Proposition 2, is strictly positive.

From the respective Euler equations (10a) and (25a) we obtain the condition that

$$r-\frac{\xi}{\sigma^R} = \rho^R > \rho^H = r - \frac{1}{\sigma^H}\left[\frac{\Delta c(t)}{c(t)}+v+\xi\right] \tag{A.26}$$

$$\Leftrightarrow \frac{\sigma^H}{\sigma^R} < \frac{1}{\xi}\left[\frac{\Delta c(t)}{c(t)}+v+\xi\right] \tag{A.27}$$

which is equivalent to Eq. (28).

A.5. Proof of Proposition 4

The optimization problem (31) subject to condition (32) is equivalent to a resource extraction model (or an isoperimetrical control problem). We denote consumption at time t of an individual of age a by $\mathcal{C}(a) \equiv c(t,t-a)$ and define the stock of consumption left to distribute among those older than age a by

$$y(a)=\overline{c}(t)-\int_0^a \mathcal{C}(a')\gamma\exp[-\nu a']\,da'. \tag{A.28}$$

Then, the problem of optimally distributing between the age groups is equivalent to optimally 'extracting' the consumption stock over age (instead of time). The equation of motion of the stock is $dy/da=-\mathcal{C}(a)\gamma\exp[-\nu a]$, the terminal condition is $y(T)\geq 0$, and the present value Hamiltonian reads

$$\mathcal{H}=\frac{\mathcal{C}(a)^{1-(1/\sigma^H)}}{1-(1/\sigma^H)}\gamma\exp[(\rho^S-\rho^H-\nu)a]-\lambda(a)\mathcal{C}(a)\gamma\exp[-\nu a], \tag{A.29}$$

where $\lambda(a)$ denotes the co-state variable of the stock y. The first order conditions yield

$$\lambda(a)=\mathcal{C}(a)^{-1/\sigma^H}\exp[(\rho^S-\rho^H)a], \tag{A.30a}$$

$$\dot{\lambda}(a)=0, \tag{A.30b}$$

which imply that

$$\mathcal{C}(a)=\mathcal{C}(0)\exp[\sigma^H(\rho^S-\rho^H)a]. \tag{A.31}$$

As $\lambda(T)$ is obviously not zero, transversality implies that $y(T)=0$. Therefore, we obtain from Eq. (A.28), acknowledging $Q_T(\nu)=1/\gamma$,

$$\mathcal{C}(0)=\overline{c}(t)\ \frac{Q_T(\nu)}{Q_T(\nu+\sigma^H(\rho^H-\rho^S))}, \tag{A.32}$$

which, together with Eq. (A.31), returns Eq. (33).

A.6. Proof of Proposition 5

(i) The equivalence of the unconstrained social planner problem and of the optimization problem in the ILA economy pointed out in relation to Eqs. (34) and (35) implies the Euler equation of the unconstrained social planner economy

$$\frac{\dot{c}(t)}{c(t)}=\sigma^H[r(t)-\rho^S]-\xi. \tag{A.33}$$

For both economies the Euler equation implies that a time varying consumption rate also implies a time varying interest rate (and obviously so does a time varying capital stock).

For observational equivalence to hold, consumption and interest rate of the unconstrained utilitarian OLG economy have to coincide with that of the ILA economy, implying the following equality of the Euler equations

$$\begin{aligned}&\sigma^H[r(t)-\rho^S]-\xi=\sigma^R[r(t)-\rho^R]-\xi\\ &\Leftrightarrow \sigma^R\rho^R-\sigma^H\rho^S=(\sigma^R-\sigma^H)r(t).\end{aligned} \tag{A.34}$$

For a time varying interest rate this equation can only be satisfied if $\sigma^R=\sigma^H$ and $\rho^H=\rho^S$.

If $\sigma^R=\sigma^H$ and $\rho^H=\rho^S$ hold, the equivalence of the two problems was explained in relation to Eqs. (34) and (35).

(ii) Existence of an observationally equivalent ILA economy implies that, first, the ILA economy has to be in a steady state as well and, second, that the steady state Euler equations have to coincide implying

$$\begin{aligned}&r=\rho^R-\frac{\xi}{\sigma^R}=\rho^S-\frac{\xi}{\sigma^H}\\ &\Rightarrow \rho^R-\rho^S=\xi\ \frac{\sigma^R-\sigma^H}{\sigma^R\sigma^H}.\end{aligned}$$

The same reasoning applies when starting from the ILA economy steady state and assuming an observationally equivalent unconstrained utilitarian OLG economy.

If Eq. (36) is satisfied and the unconstrained utilitarian OLG economy is in a steady state, Eq. (A.33) implies

$$r^S=\rho^S+\frac{\xi}{\sigma^H}. \tag{A.35}$$

Using Eq. (36) to substitute ρ^S on the right hand side yields

$$r^S=\rho^R-\xi\ \frac{\sigma^R-\sigma^H}{\sigma^R\sigma^H}+\frac{\xi}{\sigma^H}=\rho^R+\frac{\xi}{\sigma^R}=r^R. \tag{A.36}$$

Thus, also the ILA economy is in a steady state (see Section 4) with coinciding interest rate. As the interest rates coincide, so does the capital stock and so do the consumption paths. Starting with the ILA steady state with interest rate r^R yields a coinciding unconstrained utilitarian OLG steady state by the same procedure.

A.7. Proof of Proposition 6

(i) According to the proof of Proposition 5, the Euler equation of the unconstrained social planner solution is (A.33). In a steady state with interest rate $r^\star$ it is satisfied for any (obviously non-empty) set of preference parameters σ^H and ρ^S satisfying

$$\rho^S + \frac{\xi}{\sigma^H} = r^\star. \tag{A.37}$$

Moreover, by virtue of Proposition 2, $\rho^S = r^\star - \xi/\sigma^H > \rho^H$ holds. Note that for all decentralized economies $\Gamma^\star r^\star < r^{gr}$. Hence, the same reasoning as in the proof of Proposition 3 can be applied to make sure that the budget constraints of the decentralized OLG and the unconstrained utilitarian social planner OLG coincide. The condition $r^\star < r^{gr}$ also implies that the social planner's transversality condition is satisfied.

(ii) Using (33), we can write the intratemporal allocation of consumption across the generations alive in steady state in the unconstrained utilitarian OLG as

$$c_S^\star(a) = \frac{c(t,t-a)}{\exp[\xi t]} = c^\star \frac{Q_T(v)}{Q_T(v+\sigma^H(\rho^H-\rho^S))} \exp[-\sigma^H(\rho^H-\rho^S)a]. \tag{A.38}$$

The intratemporal allocation of consumption in the decentralized OLG economy is given by (14a) and can be written as

$$c_d^\star(a) = c^\star \frac{Q_T(v)}{Q_T(v+\xi-\sigma^H(r_d^\star-\rho^H))} \exp[(\sigma^H(r_d^\star-\rho^H)-\xi)a], \tag{A.39}$$

where $r_d^\star$ is the steady state interest rate of the decentralized OLG in which the households exhibit the same preference parameters as in the unconstrained utilitarian OLG economy.

⇒: Suppose that the allocation of consumption across all generations alive at each point is identical. For this to be the case, the following two equations have to hold simultaneously for all $a \in [0,T]$

$$\exp[-\sigma^H(\rho^H-\rho^S)a] = \exp[(\sigma^H(r_d^\star-\rho^H)-\xi)a], \tag{A.40a}$$

$$\sigma^H(\rho^H-\rho^S) = \xi - \sigma^H(r_d^\star-\rho^H). \tag{A.40b}$$

Minor mathematical transformations show that this only holds for

$$\rho^S = r_d^\star - \frac{\xi}{\sigma_H}. \tag{A.41}$$

This is the condition for the unconstrained utilitarian OLG and the decentralized OLG to be observationally equivalent in steady state.

⇐: Now suppose that the unconstrained utilitarian OLG and the decentralized OLG are observationally equivalent in steady state, i.e., Eq. (A.41) is satisfied.

Inserting ρ^S as given by (A.41) into (A.38) yields

$$c_S^\star(a) = c^\star \frac{Q_T(v)}{Q_T(v+\xi-\sigma^H(r_d^\star-\rho^H))} \exp[(\sigma^H(r_d^\star-\rho^H)-\xi)a], \tag{A.42}$$

which is identical to (A.39). Hence, observational equivalence in steady state is also sufficient for identical allocations across the generations alive in both economies.

A.8. Proof of Proposition 7

We show that the constrained social planner can implement the steady state social optimum with a tax/subsidy regime on interest and wages only if the steady states of the first-best optimum and the decentralized OLG economy coincide. This implies that the first-best solution is, in general, not implementable, as every first-best solution converges to a non-implementable steady state.

We show that for a given steady state, the intratemporal distribution of consumption coincides in the constrained and the unconstrained utilitarian OLG economy if and only if $\tau_r^\star = 0$. To see this consider an unconstrained utilitarian OLG economy in steady state. The household problem in the constrained utilitarian OLG economy is identical to the household problem in the decentralized economy if we substitute $r(t)$ by $r^e(t)$ and $w(t)$ by $w^e(t)$. Solving for individual consumption and wealth in the steady states yields analogously to Eqs. (14a) and (14b):

$$c^{e\star}(a) \equiv \left.\frac{c^e(t,s)}{\exp[\xi t]}\right|_{k=k^\star} = w^{e\star} \frac{Q_T(r^{e\star}-\xi)}{Q_T(r^{e\star}-\sigma^H(r^{e\star}-\rho^H))} \exp[(\sigma^H(r^{e\star}-\rho^H)-\xi)a], \tag{A.43a}$$

$$b^{e\star}(a) \equiv \left.\frac{b^e(t,s)}{\exp[\xi t]}\right|_{k=k^\star} = w^{e\star} Q_a(r^{e\star}-\sigma^H(r^{e\star}-\rho^H)) \exp[(r^{e\star}-\xi)a]\left[\frac{Q_a(r^{e\star}-\xi)}{Q_a(r^{e\star}-\sigma^H(r^{e\star}-\rho^H))} - \frac{Q_T(r^{e\star}-\xi)}{Q_T(r^{e\star}-\sigma^H(r^{e\star}-\rho^H))}\right], \tag{A.43b}$$

where $r^{e\star} = r^e(t)$ and $w^{e\star} = w^e(t)/\exp[\xi t]$, both evaluated at the steady state. Following the aggregation rule (9), we derive for aggregate steady state consumption and wealth:

$$c^{e\star} = w^{e\star} \frac{Q_T(r^{e\star}-\xi)}{Q_T(\nu)} \frac{Q_T(\nu+\xi-\sigma^H(r^{e\star}-\rho^H))}{Q_T(r^{e\star}-\sigma^H(r^{e\star}-\rho^H))}, \tag{A.44a}$$

$$b^{e\star} = \frac{w^{e\star}}{r^{e\star}-\xi}\left[\frac{Q_T(\xi+\nu-r^{e\star})}{Q_T(\nu)}-1\right] - \frac{w^{e\star}}{r^{e\star}-\sigma^H(r^{e\star}-\rho^H)} \frac{Q_T(r^{e\star}-\xi)}{Q_T(\nu)} \frac{Q_T(\xi+\nu-r^{e\star})-Q_T(\xi+\nu-\sigma^H(r^{e\star}-\rho^H))}{Q_T(r^{e\star}-\sigma^H(r^{e\star}-\rho^H))}. \tag{A.44b}$$

Inserting Eq. (A.44a) into Eq. (A.43a), we obtain the following intratemporal distribution of consumption

$$c^{e\star}(a) = c^{e\star} \frac{Q_T(\nu)}{Q_T(\nu+\xi-\sigma^H(r^{e\star}-\rho^H))} \exp\left[(\sigma^H(r^{e\star}-\rho^H)-\xi)a\right]. \tag{A.45}$$

By virtue of Eq. (33), however, the steady state intertemporal distribution of consumption in the social optimum yields:

$$c^{\star}(a) = c^{\star} \frac{Q_T(\nu)}{Q_T(\nu-\sigma^H(\rho^S-\rho^H))} \exp[(\sigma^H(\rho^S-\rho^H))a]. \tag{A.46}$$

Aggregate equivalence requires that $c^{e\star} = c^{\star}$. Distributional equivalence at a point in time requires moreover that Eq. (A.45) and Eq. (A.46) coincide. Together these conditions imply that $\sigma^H(r^{e\star}-\rho^H)-\xi = \sigma^H(\rho^S-\rho^H) \Leftrightarrow r^{e\star} = \rho^S + \frac{\xi}{\sigma^H}$. Thus, by Eq. (A.37), it must be $r^{e\star} = r^{\star}$ and therefore $\tau_r^{\star} = 0$.

A.9. *Characteristics of the functions characterizing the steady state capital stock*

Lemma 3.
The function $Q_T(r)$ defined in (13) satisfies:

(i) $Q_T(r) > 0$ *for all* $r \in \mathbb{R}$,
(ii) $Q_T{}'(r) < 0$ *for all* $r \in \mathbb{R}$.

The function

$$q(r) \equiv \frac{Q_T{}'(r)}{Q_T(r)} = \frac{T}{\exp(rT)-1} - \frac{1}{r}, \tag{A.47}$$

satisfies

(iii) $q(r) < 0$ *for all* $r \in \mathbb{R}$,
(iv) $\lim_{r\to\infty} q(r) = 0$ *and* $\lim_{r\to-\infty} q(r) = -T$,
(v) $q'(r) = q'(-r) > 0$ *for all* $r \in \mathbb{R}$,
(vi) $q'(r) > z^2 q'(zr)$ *for all* $r \in \mathbb{R}$, $z \in (0,1)$,
(vii) $y^2 q'(yr) > z^2 q'(zr)$ *for all* $r \in \mathbb{R}$, $y > z \geq 1$,
(viii) $q''(r) < 0$ *for all* $r \in \mathbb{R}_{++}$.

Proof. (i) Obviously, $Q_T(r) > 0$ for all $r \neq 0$. In addition, $\lim_{r\to 0} Q_T(r) = T > 0$.
(ii) We obtain

$$Q_T{}'(r) = -\frac{1-\exp[-rT](1+rT)}{r^2}.$$

For all $r \neq 0$:

$$Q_T{}'(r) < 0 \Leftrightarrow \exp[-rT](1+rT) < 1 \Leftrightarrow 1+rT < \exp[rT].$$

The last inequality holds as $x+1 < \exp[x]$ for all $x \in \mathbb{R}$. In addition, $\lim_{r\to 0} Q_T{}'(r) = -T^2/2 < 0$.
(iii) Follows directly from items (i) and (ii).
(iv) Follows directly from the definition (A.47).
(v) We obtain:

$$q'(r) = -\frac{1}{r^2} - \frac{T^2 \exp[-rT]}{(1-\exp[-rT])^2} = \frac{1}{r^2} - \frac{T^2}{2(\cosh[rT]-1)}.$$

For all $r \neq 0$:

$$q'(r) > 0 \Leftrightarrow 2(\cosh[rT]-1) > r^2 T^2 \Leftrightarrow \cosh[rT] > 1 + \frac{r^2 T^2}{2}.$$

The last inequality holds as $\cosh[x] > 1 + x^2/2$ for all $x \in R$. In addition, $\lim_{r\to 0} q'(r) = \frac{T^2}{12} > 0$.

(vi) The statement holds if and only if:

$$q'(r)-z^2q'(zr)=\frac{z^2\,T^2}{2(\cosh[zrT]-1)}-\frac{T^2}{2(\cosh[rT]-1)}>0\Leftrightarrow z^2(\cosh[rT]-1)>\cosh[zrT]-1.$$

To see that the last inequality holds, we employ the infinite series expansion of $\cosh[x]$:

$$z^2(\cosh[x]-1)-(\cosh[zx]-1)=z^2\left(\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}-1\right)-\left(\sum_{n=0}^{\infty}\frac{(zx)^{2n}}{(2n)!}-1\right)$$
$$=z^2\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}-\sum_{n=1}^{\infty}\frac{(zx)^{2n}}{(2n)!}=\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}(z^2-z^{2n})>0.$$

The inequality holds, as the first summand is zero and all other terms are strictly positive for all $z\in(0,1)$.

(vii) The statement holds if and only if:

$$y^2q'(yr)-z^2q'(zr)=\frac{z^2\,T^2}{2(\cosh[zrT]-1)}-\frac{y^2\,T^2}{2(\cosh[yrT]-1)}>0$$
$$\Leftrightarrow z^2(\cosh[yrT]-1)>y^2\cosh[zrT]-1.$$

Employing the infinite series expansion of $\cosh[x]$, we obtain

$$z^2(\cosh[yx]-1)-y^2(\cosh[zx]-1)=z^2\left(\sum_{n=0}^{\infty}\frac{(yx)^{2n}}{(2n)!}-1\right)-y^2\left(\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}-1\right)$$
$$=z^2\sum_{n=1}^{\infty}\frac{(yx)^{2n}}{(2n)!}-y^2\sum_{n=1}^{\infty}\frac{(yx)^{2n}}{(2n)!}=\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}z^2y^2\left(y^{2(n-1)}-z^{2(n-1)}\right)>0.$$

The inequality holds, as the first summand is zero and all other terms are strictly positive for all $y>z\geq 1$.

(viii) We obtain:

$$q''(r)=-\frac{2}{r^3}+\frac{2T^3\sinh[rT]}{(2\cosh[rT]-2)^2}=-2T^3\left(\frac{1}{(rT)^3}+\frac{\sinh[rT]}{(2\cosh[rT]-2)^2}\right)$$

Then, the statement holds if and only if $(\cosh[x]-2)^2>x^3\sinh[x]$. To see this, we employ the infinite series expansion of $\cosh[x]$ and $\sinh[x]$

$$\left(2\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}-2\right)^2-x^3\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}=\left(2\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}\right)^2-\sum_{n=0}^{\infty}\frac{x^{2n+4}}{(2n+1)!}=4\left(\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}\right)^2-\sum_{n=0}^{\infty}\frac{x^{2n+4}}{(2n+1)!}$$

Both series exhibit all even powers of x starting with x^4:

$$x^4\left(\frac{4}{2!2!}-1\right)+x^6\left(\frac{2\cdot 4}{2!4!}-\frac{1}{3!}\right)+x^8\left(\frac{2\cdot 4}{2!6!}+\frac{4}{4!4!}-\frac{1}{5!}\right)+\cdots\geq 0.$$

The inequality holds as the first term is zero and all other terms are strictly positive for all $x\in\mathbb{R}_{++}$. □

Lemma 4.
For all $\xi,v\in\mathbb{R}_{++}$, the function J defined in (A.2) satisfies

(i) $J(r)>0$.

For all $\xi,v\in\mathbb{R}_{++}$ and $\sigma^H\in(0,1]$, the function J satisfies

(ii) $d/dr(J'(r)/J(r))>0$ *for all* $r\geq\xi$,
(iii) $\lim_{r\to\infty}J'(r)/J(r)=\sigma^H T$.
For all $\xi,v\in\mathbb{R}_{++}$ and $\sigma^H>1$, the function J satisfies

(iv) $d/dr(J'(r)/J(r))>0$ *for all* $r\geq v+\xi$ *and* $\rho^H<\sigma^H-1/\sigma^H(v+\xi)$,
(v) $\lim_{r\to\infty}J'(r)/J(r)=T$.

Proof. (i) Follows immediately from $Q_T(r)>0$ for all $r\in\mathbb{R}$ as shown in Lemma 3.

(ii) Using the definition (A.47), we obtain

$$\frac{J'(r)}{J(r)} = q(r-\xi)-\sigma^H q(v+\xi-\sigma^H(r-\rho^H))-(1-\sigma^H)q(r-\sigma^H(r-\rho^H)), \tag{A.48a}$$

and

$$M(r) \equiv \frac{d}{dr}\left(\frac{J'(r)}{J(r)}\right) = \frac{J''(r)}{J(r)} - \left(\frac{J'(r)}{J(r)}\right)^2 = q'(r-\xi)+(\sigma^H)^2 q'(v+\xi-\sigma^H(r-\rho^H))-(1-\sigma^H)^2 q'(r-\sigma^H(r-\rho^H)). \tag{A.48b}$$

For $\sigma^H \in (0,1]$ set $x = r-\xi$ and restrict attention to all $x \geq 0$

$$\begin{aligned} M(x) &= q'(x)+(\sigma^H)^2 q'(v+(1-\sigma^H)\xi-\sigma^H(x-\rho^H))-\left(1-\sigma^H\right)^2 q'((1-\sigma^H)x+(1-\sigma^H)\xi+\sigma^H\rho^H) \\ &> q'(x)-(1-\sigma^H)^2 q'((1-\sigma^H)x+(1-\sigma^H)\xi+\sigma^H\rho^H) \geq q'(x)-(1-\sigma^H)^2 q'((1-\sigma^H)x) \geq 0. \end{aligned}$$

The first inequality holds due to part (v), the second inequality due to part (viii) and the last inequality due to part (vi) of Lemma 3.

(iii) Follows directly from Eq. (A.48a) and part (iv) of Lemma 3.

(iv) For $\sigma^H > 1$ and $\rho^H < (\sigma^H-1)/\sigma^H(v+\xi)$ consider only $r \geq v+\xi$

$$\begin{aligned} M(r) &= q'(r-\xi)+(\sigma^H)^2 q'(\sigma^H r-\sigma^H\rho^H-(v+\xi))-(\sigma^H-1)^2 q'((\sigma^H-1)r+\sigma^H r) \\ &> (\sigma^H)^2 q'(\sigma^H r-\sigma^H\rho^H-(v+\xi))-(\sigma^H-1)^2 q'((\sigma^H-1)r+\sigma^H r) > (\sigma^H)^2 q'(\sigma^H r)-(\sigma^H-1)^2 q'((\sigma^H-1)r) \geq 0 \end{aligned}$$

The first inequality holds due to part (v), the second inequality due to part (viii) and the last inequality due to part (vii) of Lemma 3.

(v) Follows directly from Eq. (A.48a) and part (iv) of Lemma 3. □

References

Aiyagari, S.R., 1985. Observational equivalence of the overlapping generations and the discounted dynamic programming frameworks for one-sector growth. Journal of Economic Theory 35, 201–221.
Andersen, S., Harrison, G.W., Lau, M.I., Rutstroem, E.E., 2008. Eliciting risk and time preferences. Econometrica 76, 583–618.
Asheim, G., Mitra, T., 2010. Sustainability and discounted utilitarianism in models of economic growth. Mathematical Social Science 59, 148–169.
Bansal, R., Yaron, A., 2004. Risks for the long run: a potential resolution of asset pricing puzzles. Journal of Finance 59, 1481–1509.
Bansal, R., Kiku, D., Yaron, A., 2010. Long-run risks, the macro-economy and asset prices. American Economic Review: Papers & Proceedings 100, 542–546.
Barro, R.J., 1974. Are government bonds net wealth? Journal of Political Economy 82, 1095–1117.
Becker, G.S., 1974. A theory of social interactions. Journal of Political Economy 82, 1063–1093.
Blanchard, O.J., 1985. Debt, deficits, and finite horizons. Journal of Political Economy 93, 223–247.
Broome, J., Schmalensee, R., 1992. Counting the Cost of Global Warming. White Horse Press, Cambridge.
Browning, M., Hansen, L.P., Heckman, J.J., 1999. Micro data and general equilibrium models. In: Taylor, J.B., Woodford, M. (Eds.), Handbook of Macroeconomics, vol. 1. , Elsevier, Amsterdam, pp. 543–633. chapter 8.
Burton, P.S., 1993. Intertemporal preferences and intergenerational equity considerations in optimal resource harvesting. Journal of Environmental Economics and Management 24, 119–132.
Calvo, G.A., Obstfeld, M., 1988. Optimal time-consistent fiscal policy with finite lifetimes. Econometrica 56, 411–432.
Campbell, J.Y., 1996. Understanding risk and return. Journal of Political Economy 104, 298–345.
Chen, X., Favilukis, J., Ludvigson, S.C., 2011. An Estimation of Economic Models with Recursive Preferences. NBER Working Paper 17130.
Cline, W.R., 1992. The Economics of Global Warming. Institute for International Economics, Washington DC.
d'Albis, H., 2007. Demographic structure and capital accumulation. Journal of Economic Theory 132, 411–434.
Epstein, L.G., Zin, S.E., 1991. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: an empirical analysis. Journal of Political Economy 99, 263–286.
Gan, Z., Lau, S.-H.P., 2010. Demographic structure and overlapping generations: a simpler proof with more general conditions. Journal of Mathematical Economics 46, 311–319.
Gerlagh, R., van der Zwaan, B.C.C., 2000. Overlapping generations versus infinitely-lived agent: the case of global warming. In: Howarth, R., Hall, D. (Eds.), The long-term economics of climate change, vol. 3. , JAI Press, Stamford, pp. 301–327.
Hall, R., 1988. Intertemporal substitution in consumption. Journal of Political Economy 96, 339–357.
Harrison, G.W., Lau, M.I., Rutstroem, E.E., Sullivan, M.B., 2005. Eliciting risk and time preferences using field experiments: some methodological issues. In: Carpenter, J., Harrison, W., List, J.A. (Eds.), Field Experiments in Economics, vol. 10. , JAI Press, Greenwich, CT, pp. 125–218.
Harrod, R.F., 1948. Towards a Dynamic Economics. Macmillan, London.
Heinzel, C., Winkler, R., 2011. Distorted time preferences and time-to-build in the transition to a low-carbon energy industry. Environmental and Resource Economics 43, 217–241.
Howarth, R.B., 1998. An overlapping generations model of climate-economy interactions. Scandinavian Journal of Economics 100, 575–591.
Howarth, R.B., 2000. Climate change and the representative agent. Environmental and Resource Economics 15, 135–148.
Hurd, M.D., 1987. Savings of the elderly and desired bequests. American Economic Review 77, 298–312.
Hurd, M.D., 1989. Mortality risk and bequests. Econometrica 57, 779–814.
Jackson, M.O., Yariv, L., 2011. Collective dynamic choice: the necessity of time inconsistency. Working Paper, Stanford Econ. Dept.
Koopmans, T.C., 1965. On the concept of optimal economic growth. In: Study Week on the Econometric Approach to Development Planning, Pontificiae Academiae Scientiarum Scripta Varia, vol. 28. North Holland, Amsterdam, pp. 225–266.
Kopczuk, W., Lupton, J.P., 2007. To leave or not to leave: the distribution of bequest motives. Review of Economic Studies 74, 207–235.
Laitner, J., Juster, F.T., 1996. New evidence on altruism: a study of TIAA-CREF retirees. American Economic Review 86, 893–908.
Laitner, J., Ohlsson, H., 2001. Bequest motives: a comparison of Sweden and the United State. Journal of Public Economics 79, 205–236.
Marini, G., Scaramozzino, P., 1995. Overlapping generations and environmental control. Journal of Environmental Economics and Management 29, 64–77.
Nordhaus, W., 1994. Managing the Global Commons. MIT Press, Cambridge.
Nordhaus, W., 2008. A Question of Balance: Economic Modeling of Global Warming. Yale University Press, New Haven.
Nordhaus, W.D., 2007. A review of the Stern review on the economics of climate change. Journal of Economic Literature 45, 686–702.

Pigou, A.C., 1932. The Economics of Welfare. Macmillan, London.
Ramsey, F., 1928. A mathematical theory of savings. Economic Journal 38, 543–559.
Sen, A., Williams, B. (Eds.), 1982. Utilitarianism and Beyond. Cambridge University Press, Cambridge and New York.
Solow, R.M., 1974. Intergenerational equity and exhaustible resources. Review of Economic Studies 41, 29–45.
Stephan, G., Müller-Fürstenberger, G., Previdoli, P., 1997. Overlapping generations or infinitely-lived agents. Intergenerational altruism and the economics of global warming. Environmental and Resource Economics 10, 27–40.
Stern, N., 2007. The Economics of Climate Change – The Stern Review. Cambridge University Press, Cambridge.
Vissing-Jørgensen, A., Attanasio, O.P., 2003. Stock-market participation, intertemporal substitution, and risk-aversion. American Economic Review 93, 383–391.
von Below, D., 2012. Optimal carbon taxes with social and private discounting. Paper Presented at the SURED Conference in Ascona, Switzerland.
Wilhelm, M.O., 1996. Bequest behavior and the effect of heirs' earnings: testing the altruistic model of bequests. American Economic Review 86, 874–892.
Yaari, M.E., 1965. Uncertain lifetime, life insurance, and the theory of the consumer. Review of Economic Studies 32, 137–150.