

University of California, Berkeley
Department of Agricultural &
Resource Economics

CUDARE Working Papers

<i>Year 2012</i>	<i>Paper 1092R3</i>
<i>Year 2012</i>	<i>Paper 1092R2</i>
<i>Year 2011</i>	<i>Paper 1092R</i>
<i>Year 2008, first version</i>	<i>Paper 1092</i>

Why Uncertainty Matters - Discounting under Intertemporal Risk Aversion and Ambiguity

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CUDARE Working Paper 1092

First version: December 2008 – This version: April 2012

Abstract: Uncertainty has an almost negligible impact on project value in the standard economic model. I show that a comprehensive evaluation of uncertainty and uncertainty attitude changes this picture fundamentally. The illustration of this result relies on the discount rate, which is the crucial determinant in balancing immediate costs against future benefits, and the single most important determinant of optimal mitigation policies in the integrated assessment of climate change. First, the paper removes an implicit assumption of (intertemporal or intrinsic) risk neutrality from the standard economic model. Second, the paper introduces aversion to non-risk uncertainty (ambiguity). I show a close formal similarity between the model of intertemporal risk aversion, which is a reformulation of the widespread Epstein-Zin-Weil model, and a recent model of smooth ambiguity aversion. I merge the models, achieving a threefold disentanglement between, risk aversion, ambiguity aversion, and the propensity to smooth consumption over time.

JEL Codes: D61, D81, D90, H43, Q00, Q54

Keywords: ambiguity, climate change, cost benefit analysis, discounting, intertemporal substitutability, risk aversion, uncertainty

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*I am grateful to Larry Karp, David Anthoff, Christian Gollier, Rick van der Ploeg, Geir Asheim, Sverre Jensen, and to participants of the EAERE 2009, the EEA 2009, and of workshops in Oslo and Santa Barbara in 2010.

1 Introduction

1.1 Overview

The paper makes three contributions to theory and shows how these contributions affect policy analysis. First, I show a close formal similarity between a model disentangling risk aversion from a decision maker's propensity to smooth consumption over time, and a model disentangling risk attitude from ambiguity attitude. Second, I merge the two models into a framework that achieves a three-fold disentanglement of Arrow-Pratt risk aversion, intertemporal substitutability, and smooth ambiguity aversion. Third, in an analytically tractable setting, I derive the resulting consumption discount rates. I show that the formal similarity between the models translates into two fully symmetric effects on the discount rate. The applied contribution of the paper illustrates the importance of a comprehensive uncertainty attitude in long-term evaluation. In the standard discounted expected utility model, uncertainty has an almost negligible impact on project value. I show how a model of comprehensive risk and uncertainty attitude changes this result. I contribute to the social discounting debate and provide simple analytic discounting formulas that pay particular attention to the correlation between growth uncertainty and project payoff.

I disentangle intertemporal consumption smoothing and risk attitude in a reformulated version of the Epstein-Zin-Weil model (Epstein & Zin 1989, Weil 1990). The reformulation builds on Traeger's (2010a,2012) measure of intertemporal risk attitude. Risk aversion has two effects on evaluation. First, stochasticity generates wiggles in the consumption path. Agents with a propensity to smooth consumption over time dislike these wiggles. Second, agents intrinsically dislike risk because it creates uncertainty over the future. Intertemporal risk aversion measures this second effect. In comparison, Arrow-Pratt risk aversion in the Epstein-Zin-Weil model measures both risk effects jointly. Expressing the Epstein-Zin-Weil model in terms of intertemporal risk aversion reveals its formal similarity to the smooth ambiguity model by Klibanoff, Marinacci & Mukerji (2005). Ambiguity models capture the distinction between uncertainty that is characterized by a unique probability distribution (risk), and more general uncertainty, usually referred to as ambiguity, hard uncertainty, deep uncertainty, or Knightian uncertainty. The similarity between the reformulated Epstein-Zin model and the smooth ambiguity model translates into equivalent terms in the consumption discount rate. These terms are proportional to intertemporal risk aversion in the Epstein-Zin setting, and proportional to ambiguity aversion in the smooth ambiguity model.

The consumption discount rate determines the optimal trade-off between current

investment costs and future investment payoffs. In the context of public projects, this consumption discount rate is known as the social discount rate. The U.K. and France have explicitly adapted their discounting schemes for the evaluation of legislation and long-term projects to recognize uncertainty. The U.S. Environmental Protection Agency is currently preparing a similar proposal to the Office of Management and Budget. The economic insights underlying these political reforms are based on the standard economic model. However, it is well-known that this standard model gives rise to a variety of puzzles in asset pricing under uncertainty, including the equity premium and the risk free rate puzzles. These puzzles are easily explained using more comprehensive uncertainty models (Basal & Yaron 2004). In particular, the finance literature shows and exploits the fact that agents are more Arrow-Pratt risk averse than they are averse to consumption substitution in the time dimension: they are intertemporal risk averse. I illustrate the importance of modeling general uncertainty attitude when incorporating uncertainty into public evaluation.

My derivation and discussion of the discount rate pays special attention to the correlation between growth uncertainty and project payoffs. This contribution has important implications for long-term evaluation. Examples of large scale projects (or legislation) with uncertain payoffs include investments into basic research, national defense, development of new energy technologies, or climate change adaptation and mitigation. I focus on the climate policy example, where the social discount rate is currently hotly debated (Stern 2007, Nordhaus 2007, Weitzman 2007, Weitzman 2009, Dasgupta 2009, Heal 2009). The social discount rate is the single most important explanatory variable, when comparing the policy recommendations resulting from different integrated assessments of climate change (Nordhaus 2007). For example, a social discount rate of 1.4%, as chosen in the Stern (2007) review, implies an optimal present day carbon tax that is 10 times higher than using a rate of 5.5%, as chosen by Nordhaus (2008).¹ First, I illustrate that thin-tailed growth risk easily results in adjustments of the social discount rate in the same order of magnitude as the difference between the rates chosen by Nordhaus and Stern. This finding contributes to a recent discussion on the importance of uncertainty in climate change evaluation (Weitzman 2009, Pindyck 2009, Nordhaus 2012). Second, in the climate context, there is support for positive, for negative, and for no correlation between mitigation payoffs and economic growth. I illustrate the major relevance of this correlation for climate policy evaluation. Thus, the paper calls for a more careful

¹Almost all large scale integrated assessment models deriving optimal policies are based on a representative agent employing the standard economic model. In regional models, like the Nordhaus (2011) RICE model, each regions is represented by such a representative agent.

analysis of the correlation channels in the climate-economy interaction.

1.2 Related Literature

The disentanglement of risk aversion from the propensity to smooth consumption over time goes back to Selden (1978), Kreps & Porteus (1978), Epstein & Zin (1989), and Weil (1990). Traeger (2010a,2012) extends the Epstein-Zin-Weil disentanglement to a multi-commodity setting. For this purpose, he introduces the concept of intertemporal risk aversion. Intertemporal risk aversion directly measures intrinsic aversion to risk, which is not captured in the standard model. Thus, discounting contributions proportional to intertemporal risk aversion measure the deviations from the discount rate in the standard model. As I point out in this paper, smooth ambiguity aversion by Klibanoff et al. (2005) and Klibanoff, Marinacci & Mukerji (2009) similarly captures a form of intrinsic aversion to ambiguity. I show that the measure of smooth ambiguity aversion is an analogue to the measure of intertemporal risk aversion, applied to ambiguous settings.

Gollier (2002) discusses the socially optimal discount rate for public investment projects in a model that disentangles risk aversion from intertemporal consumption smoothing. He identifies a sufficient condition for discount rates to be lower under uncertainty than under certainty: the disentangled Arrow Pratt measure of absolute risk aversion has to decrease in consumption. This finding closely relates to Leland's (1968) finding in the standard model: savings increase under uncertainty if entangled Arrow Pratt risk aversion decreases in consumption. This condition is widely believed to hold and, in particular, it is satisfied in case of isoelastic preferences employed in the current paper's application. Apart from adding ambiguity, the present paper complements and extends Gollier's (2002) analysis in several ways. First, I derive an exact formula that differs from Gollier's approximate formula for the social discount rate under isoelastic preferences. Second, I use quantitative estimates from the asset pricing literature to discuss the magnitude of the various contributions to the social discount rate. Third, my formulation extracts the difference in discounting between the general and the standard economic model into a single simple-to-interpret adjustment of the discount rate, proportional to intertemporal risk aversion. Fourth, this reformulation in terms of intertemporal risk aversion shows that Epstein-Zin preferences and smooth ambiguity aversion imply largely equivalent adjustments of the discount rate. Fifth, I extend the setting to account for the important correlation between project payoffs and baseline uncertainty.

Paralleling the current research, Gierlinger & Gollier (2008) analyze the social dis-

count rate in Klibanoff et al.'s (2005) smooth ambiguity framework. While I focus on the analytic extension of the Ramsey formula and discuss the magnitude and relevance of individual terms, Gierlinger & Gollier (2008) focus on general ambiguity attitude and qualitative characterizations of the impact of uncertainty and ambiguity aversion. In comparison to their paper, I sacrifice generality of functional forms for the sake of analytic tractability and generality in terms of conceptual extensions. First, the current paper adds stochasticity of the investment projects and shows how general uncertainty attitude amplifies the importance of the correlation between economic baseline growth and the stochastic payoffs of the project. Second, Klibanoff et al. (2005) and, thus, Gierlinger & Gollier (2008) conflate the disentanglement between the propensity to smooth consumption over time versus risk states with aversion to ambiguity.² In contrast, I provide a clear threefold disentanglement of all three preference dimensions. A minor difference is that my derivation of the social discount rate does not rely on the assumption of a rational and efficient equilibrium. In the climate change context, for example, mitigation efforts are not close to an efficient Lucas (1987) tree model equilibrium as described in Gierlinger & Gollier (2008).

Weitzman (2007, 2009) argues that uncertainty gives rise to a low social discount rate in climate change assessment. Weitzman reaches this conclusion by following a Bayesian approach to modeling structural uncertainty. His analysis delivers a fat-tailed posterior over damages that translates into a high willingness to pay for a (certain) transfer into the future. Instead of following Weitzman's path of augmenting uncertainty to a questionable level,³ I follow the decision theoretic developments that treat uncertainty attitude more comprehensively. Finally, following the original working paper version of this paper, Ju & Miao (2009) calibrated a version of the three-fold disentangling model I present here.⁴ A motivation of general risk and uncertainty attitude on the basis of observed behavior can raise a concern regarding the applicability of the current analysis

²In the original smooth ambiguity model, aversion to standard or objective risk is set equal to the propensity to smooth consumption over time. Only aversion to subjective risk, or second order uncertainty, is disentangled from this intertemporal smoothing preference. Thus, the original smooth ambiguity model conflates ambiguity aversion with well known risk characteristics: objective risk aversion is usually larger than the propensity to smooth consumption intertemporally. Introducing a disentanglement from intertemporal smoothing only for subjective uncertainty results in an unfair comparison between the effects of risk and ambiguity aversion.

³For critical discussions of this approach, see in particular Pindyck (2009), Nordhaus (2009), Horowitz & Lange (2009), and Millner (2011).

⁴They find a coefficient of smooth ambiguity aversion very close to the risk aversion coefficients I discuss in the context of risk aversion. However, their approach exogenously assumes a low value of Arrow-Pratt (and, thus, intertemporal) risk aversion. Then, their coefficient of relative smooth ambiguity aversion picks up the remaining aversion necessary to explain asset prices.

to public decision making.⁵ In response to this criticism, Traeger (2010b) provides a normative axiomatic foundation to decision making under intertemporal risk aversion and under smooth ambiguity version. Moreover, Traeger (2011) shows that the basic isoelastic discounting formulas derived here carry over to the multi-period setting.

Section 2 provides the background to the paper. It discusses the consumption discount rate in the standard model and introduces the concept of intertemporal risk aversion. Section 3 extends the discounting formula to intertemporal risk aversion and stochastic projects. It shows that, in a model of comprehensive risk attitude, even minor growth risk can reduce the social discount rate to the level of pure time preference. Section 4 incorporates smooth ambiguity aversion. It applies the model to second order uncertainty over expected growth and over the correlation between project payoff and baseline uncertainty. Section 5 concludes.

2 Background

2.1 Discounting the Future under Uncertain Growth

The consumption or social discount rate characterizes how the value of consumption develops over time. This section lays out the basic setting and summarizes important aspects of the recent debate over the correct social discount rate (Stern 2007, Nordhaus 2007, Weitzman 2007, Weitzman 2009, Dasgupta 2009, Plambeck, Hope & Anderson 1997). First period consumption $x_1 \in X$ is certain while second period consumption is captured by the probability measure p over X .⁶ In the *standard model*, a decision maker evaluates utility for every period and for every state of the world by a utility function u and sums over states and over time to arrive at

$$U^s(x_1, p) = u(x_1) + e^{-\delta} E_p u(x_2) . \quad (1)$$

The utility discount rate δ is known as the rate of pure time preference.

The decision maker faces a trade-off between aggregate consumption in the present and in the future. Growth is stochastic and his utility is $u(x) = \frac{x^{1-\eta}}{1-\eta}$, $\eta > 0, \eta \neq 1$. Given x_1 , the consumption growth rate $g = \ln \frac{x_2}{x_1}$ is normally distributed with $g \sim N(\mu, \sigma^2)$. The *risk-free social or consumption discount rate* $r = \ln \frac{dx_2}{-dx_1} |_{\bar{U}}$ characterizes a marginal

⁵In particular, some ambiguity models (but not the smooth ambiguity model) imply time inconsistent decisions, which might not be considered rational or normatively desirable.

⁶Formally, X is a compact metric space and $p \in P$ an element of the space of Borel probability measures on X .

certain trade-off between the future (dx_2) and the present (dx_1) that leaves overall welfare unchanged:

$$r = \delta + \eta\mu - \eta^2 \frac{\sigma^2}{2}. \quad (2)$$

The formula is a well-known extension of the classic Ramsey (1928) formula that makes growth stochastic. More precisely, the consumption or social discount rate equals the right hand side of the Ramsey equation. Given incomplete markets, externalities, and transaction costs, this right hand side is a preferred measure for optimality of trade-offs characterizing long-term projects and legislation that affect consumption. I emphasize the *risk-free* nature of the trade-off characterized by the consumption (or social) discount rate. The rate evaluates deterministic projects in an uncertain growth scenario. Frequently, this rate in equation (2) is also applied to evaluate certainty-equivalent project payoffs. However, section 3.2 discusses why the stochastic discount rate should be used instead for evaluating stochastic projects. The first term characterizing the discount rate in equation (2) reflects pure impatience. The second term is a consequence of economic growth. The consumption elasticity of marginal utility η characterizes the percentage decrease in marginal utility from a percentage increase of consumption. Together with the expected growth rate μ , the term $\eta\mu$ equals the decrease of marginal utility because of expected consumption growth. The parameter η captures aversion to intertemporal consumption changes. Apart from expected growth, these changes include expected wiggles that are caused by stochastic fluctuations. The decision maker's aversion to these wiggles is captured in the term $\mu \frac{\sigma^2}{2}$. This aversion to wiggles in the consumption path is the only risk contribution in the standard model and η is simultaneously interpreted as a measure of risk aversion. For the annual discount rate, the parameters δ , μ , and σ are in the order of percent, while η is in the unit order. Therefore, σ^2 makes the third term 10 – 100 times smaller than the others, and risk can be neglected in discounting.⁷

The parameter choices of Stern (2007) can be approximated by $\delta = 0.1\%$, $\eta = 1$, and $\mu = 1.3\%$ delivering $r = 1.4\%$ under certainty. While Stern's team argues from a normative perspective for these choices, the majority of integrated assessment modelers reject that standpoint.⁸ A representative of this positive school is Nordhaus, creator of the widely used open-source integrated assessment model DICE. His parameter choices

⁷The parameter σ characterizes risk in the sense of volatility. In the climate change debate, risk is frequently used to also denote a reduction in the expected value, e.g. as a consequence of catastrophic events. Such a reduction mostly affects the expected growth term of the social discount rate.

⁸Moreover, Dasgupta (2008) points out that, from a normative perspective, an egalitarian choice of $\delta = 0.1\%$ should also call for a higher propensity of intergenerational consumption smoothing $\eta > 1$.

in DICE-2007 (Nordhaus 2008) are $\delta = 1.5\%$, $\eta = 2$, and $\mu = 2\%$ ⁹ delivering $r = 5.5\%$ (again under certainty). This difference in the social discount rate implies a factor 10 difference in the resulting optimal carbon tax. Introducing uncertainty with a standard deviation of $\sigma = 2\%$ (4%) results in an adjustment of the risk-free rate by 0.02% (0.08%) in the case of Stern and 0.08% (0.3%) in the case of Nordhaus. The lower standard deviation of $\sigma = 2\%$ is used by Weitzman (2009) to approximate the volatility of economic growth without climate change and catastrophic risks. The high standard deviation of $\sigma = 4\%$ is the rounded estimate of historic consumption fluctuations by Kocherlakota (1996).¹⁰ The values for the low standard deviation are negligible, while the high standard deviation results in minimal adjustments.

2.2 Intertemporal Risk Aversion

The standard model of the previous section implicitly assumes that a decision maker's aversion to risk coincides with his aversion to intertemporal variation. Epstein & Zin (1989) and Weil (1990) derive an alternative setting that disentangles these two a priori quite different characteristics of preference. Traeger (2010a) extends their framework to a multi-commodity setting, introducing a new measure of *intertemporal risk aversion* that measures the difference between Arrow-Pratt risk aversion and the propensity to smooth consumption over time. The current section motivates the Epstein-Zin generalization of risk attitude along the lines of intertemporal risk aversion. Later sections use of the intertemporal risk aversion measure to give a more compact characterization of the social discount rate adjustment under general risk attitude and to show and exploit its similarity to the smooth ambiguity measure in the case of general uncertainty. From the perspective of intertemporal risk aversion, the standard model is risk neutral. It only generates aversion to stochasticity because of the resulting wiggles in the consumption path and a decision maker's propensity to smooth consumption over time, not because of intrinsic aversion to being uncertain about the future. As a consequence, expressing the social discount rate in terms of intertemporal risk aversion splits its constituents cleanly into those contributions arising in the standard model and those additional contributions that are due to intrinsic risk aversion.

The curvature of the utility function u in equation (1) captures both aversion to risk and aversion to intertemporal variation. A priori, however, risk aversion and the

⁹The growth rate is endogenous in the DICE model and has been reconstructed from Nordhaus (2007, 694).

¹⁰Kocherlakota (1996) estimates $\mu = 1.8\%$ and $\sigma = 3.6\%$ based on 90 years of annual data for the US.

propensity to smooth consumption over time are two distinct concepts. More generally, welfare is characterized by two independent functions corresponding to these two distinct preference characteristics

$$U(x_1, p) = u(x_1) + e^{-\delta} f^{-1} [E_p f \circ u(x_2)] . \quad (3)$$

Under certainty, the f -functions cancel. Therefore, utility u is a measure for the appreciation of consumption that derives from the willingness to trade over time. The concavity of u captures the aversion to intertemporal consumption variation. The curvature of f describes intertemporal risk aversion, which can be interpreted as aversion with respect to utility gains and losses. Note that the curvature of f is a one-dimensional risk measure even in a multi-commodity world.¹¹ Contrary to a widespread belief, equation (3) – not equation (1) – is the general representation of preferences satisfying the von Neumann & Morgenstern (1944) axioms, additive separability over certain consumption paths, time consistency, and (finite time) stationarity (Traeger 2007).¹²

A representation-free, i.e., choice based characterization of intertemporal risk aversion motivates why the standard model generally falls short of capturing risk attitude comprehensively. The general definition is provided in Traeger (2010a). It requires at least two uncertain periods. Here, I give a simplified characterization that requires the absence of pure time preference.¹³ Let $\succeq$ characterize preferences on $X \times P$ representable by equation (3) with $\delta = 0$. A decision maker is called (weakly)¹⁴ intertemporal risk averse, if and only if, for all $x^*, x_1, x_2 \in X$

$$(x^*, x^*) \sim (x_1, x_2) \quad \Rightarrow \quad (x^*, x^*) \succeq (x^*, (\tfrac{1}{2}, x_1; \tfrac{1}{2}, x_2)) , \quad (4)$$

where the term $(\frac{1}{2}, x_1; \frac{1}{2}, x_2)$ denotes a fair coin flip returning either x_1 or x_2 . The premise in equation (4) states that a decision maker is indifferent between a certain constant consumption path delivering the same outcome x^* in both periods and another certain consumption path that delivers outcome x_1 in the first and outcome x_2 in the

¹¹See Kihlstrom & Mirman (1974) for the complications that arise when trying to extend the Arrow Pratt risk measures to a multi-commodity setting. Even more interestingly, measures of intertemporal risk aversion can be applied straightforwardly to contexts where impacts do not have a natural cardinal scale.

¹²Note that, in general, preferences represented by equation (3) cannot be represented by an evaluation function of the form $U^s(x_1, p) = u_1(x_1) + E_p u_2(x_2)$.

¹³I abandon pure time preference for the sake of simplicity in the characterization only. This step does not change the intuition of the axiom with respect to its general form. Obviously, I keep pure time preference when discussing discount rates.

¹⁴The strong notion would involve the additional requirement $(x^*, x_1) \not\sim (x^*, x_2)$ in the premise and a strict preference in the implication.

second period. For example, x_1 can be an inferior outcome with respect to x^* . Then, x_2 is a superior outcome with respect to x^* . On the right-hand side of equation (4), the decision maker receives x^* in the first period, independent of his choice. For the second period, he has a choice between the certain outcome x^* or a lottery that returns with equal probability either the superior or the inferior outcome. The decision maker is called (weakly) intertemporal risk averse if he prefers the certain outcome x^* in the second period over the lottery.¹⁵ I show in Proposition 5 in Appendix B that a decision maker (with $\delta = 0$) is intertemporal risk averse in the sense of equation (4) if and only if the function f in the representation (3) is concave.

In the two period setting, the intertemporally additive reformulation of Epstein & Zin's (1989, 1991) infinite horizon recursive utility model is¹⁶

$$U(x_1, p) = \frac{x_1^{1-\eta}}{1-\eta} + e^{-\delta} \frac{1}{1-\eta} \left[E_p x_2^{1-\text{RRA}} \right]^{\frac{1-\eta}{1-\text{RRA}}}, \quad (5)$$

where RRA is the coefficient of Arrow Pratt risk aversion. It is easily verified that equation (5) results from equation (3) using the intertemporal risk aversion function

$$f(z) = ((1-\eta)z)^{\frac{1-\text{RRA}}{1-\eta}}. \quad (6)$$

Instead of Arrow-Pratt risk aversion, I will make frequent use of the measure of relative

¹⁵The lottery on the right-hand side of equation (4) will either make the decision maker better off or worse off than (x^*, x^*) , while, on the left-hand side, the decision maker knows that if he picks an inferior outcome for some period he certainly receives the superior outcome in the other.

Calling preferences satisfying equation (4) intertemporal risk averse is motivated by the facts that, first, the definition intrinsically builds on intertemporal trade-offs and, second, Normandin & St-Amour (1998, 268) make the point that the conventional Arrow Pratt measure of risk aversion is an atemporal concept.

A decision maker is defined as (weakly) intertemporal risk loving if the preference relation $\succeq$ in equation (4) is replaced by $\preceq$. He is defined to be risk neutral if he is both intertemporal risk loving and intertemporal risk averse (relation $\succeq$ in equation 4 is replaced by $\sim$).

¹⁶In a multiperiod framework equation (5) translates into the recursion

$$U(x_{t-1}, p_t) = \frac{x_{t-1}^{1-\eta}}{1-\eta} + \beta \frac{1}{1-\eta} \left[E_{p_t} ((1-\eta)U(x_t, p_{t+1}))^{\frac{1-\text{RRA}}{1-\eta}} \right]^{\frac{1-\eta}{1-\text{RRA}}}, \quad (\star)$$

To obtain the normalization used by Epstein & Zin (1989, 1991), multiply equation $(\star)$ by $(1-\beta)(1-\eta)$ and take both sides to the power of $\frac{1}{1-\eta}$. Define $U^*(x_{t-1}, p_t) = ((1-\beta)(1-\eta)U(x_{t-1}, p_t))^{\frac{1}{1-\eta}}$. Expressing the resulting transformation of equation $(\star)$ in terms of U^* delivers their version

$$U^*(x_{t-1}, p_t) = \left((1-\beta)x_{t-1}^{1-\eta} + \beta \left[E_{p_t} (U^*(x_t, p_{t+1}))^{1-\text{RRA}} \right]^{\frac{1-\eta}{1-\text{RRA}}} \right)^{\frac{1}{1-\eta}}.$$

intertemporal risk aversion

$$\text{RIRA}(z) = -\frac{f''(z)}{f'(z)} |z| = \begin{cases} 1 - \frac{1-\text{RRA}}{1-\eta} & \text{if } \eta < 1 \\ \frac{1-\text{RRA}}{1-\eta} - 1 & \text{if } \eta > 1. \end{cases} \quad (7)$$

The measure $\text{RIRA}(z)$ depends on the choice of zero in the definition of the utility function u . This normalization-dependence is the analog to e.g. the wealth level dependence of the Arrow Pratt measure of relative risk aversion.¹⁷¹⁸ Traeger (2010a) further elaborates that f and RIRA can be interpreted as a measure for the difference between Arrow Pratt risk aversion and the willingness to smooth consumption over time.

2.3 Quantification of Parameters

For my quantitative analysis, I focus on the case of risk, where parametric estimates have converged to a more reliable quantification of general attitude than for ambiguity. However, I show that the formal adjustments of the discount rate to include ambiguity attitude are almost identical to the adjustment in the case of intertemporal risk aversion. Thus, the quantitative illustration in the case of risk translates immediately into an illustration of the quantitative effects under ambiguity aversion. Estimates of the generalized isoelastic model usually build on Epstein & Zin (1991) and Campbell's (1996) log-linearizing the Euler equations. The estimation of the isoelastic model is significantly more challenging than in the case of the standard model.¹⁹ However, over the recent years, a somewhat reliable set of parameters emerges to be $\eta = \frac{2}{3}$ and $\text{RRA} \in [8, 10]$, explaining well asset prices and related puzzles (Vissing-Jørgensen & Attanasio 2003, Basal & Yaron 2004, Basal, Kiku & Yaron 2010). The message of these estimates and calibration results is that agents tend to have a higher aversion to risk than to intertemporal substitution. I denote the corresponding preference scenario by "D" for "disentangled model" and use the values $\eta = \frac{2}{3}$ and $\text{RRA} = 9.5$ that are emphasized by Vissing-Jørgensen & Attanasio (2003). These estimates imply a coefficient of

¹⁷In the standard model, the Arrow Pratt measure of relative risk aversion depends on what is considered the $x = 0$ level. For example, whether or not breathing fresh air is part of consumption or whether human capital is part of wealth changes the Arrow Pratt coefficient.

¹⁸Note that positivity of RIRA indicates intertemporal risk aversion independently of whether f is increasing and concave or decreasing and convex (see footnote 33). In both cases $-\frac{f''}{f'}$ is positive. Moreover, measuring utility in negative units as in the isoelastic case for $\rho < 0$ makes z negative. Therefore, the definition of relative risk aversion has to employ the absolute of the variable z (Traeger 2010a). The same reasoning applies to the measure of smooth ambiguity aversion.

¹⁹These models have to make assumptions about the covariance of consumption growth and stock returns, the share of stocks in the financial wealth portfolio, the properties of the expected returns to human capital, and the share of human capital in overall wealth.

relative intertemporal risk aversion of $RIRA = 27$. The standard model does not allow the decision maker to distinguish between risk and intertemporal smoothing attitude. In consequence, the joint, entangled estimate usually fall somewhere in-between the disentangled estimates of Arrow-Pratt risk aversion and the propensity to smooth consumption over time. For my quantitative analysis, I use the entangled standard model with $\eta = 2$ as scenario “N”. The value of 2 is widespread and, in particular, employed in Nordhaus’s (2008) integrated assessment of climate change. The standard model implies zero intertemporal risk aversion. Depending on the assessment, I also provide sensitivity scenarios or vary parameters on a continuum.

3 Discounting Under Intertemporal Risk Aversion

3.1 Risk Free Projects

Intertemporal risk aversion results in the following adjustment of the risk-free rate.

Proposition 1: The risk-free social/consumption discount rate in the isoelastic setting with intertemporal risk aversion is

$$r = \delta + \eta\mu - \eta^2 \frac{\sigma^2}{2} - RIRA |1 - \eta^2| \frac{\sigma^2}{2}. \quad (8)$$

In the presence of growth uncertainty, a decision maker exhibiting positive intertemporal risk aversion $RIRA > 0$ discounts the future payoffs at a lower rate. In consequence, a certain project with a relatively low productivity can still improve the welfare of an intertemporal risk averse decision maker, even if an evaluation based on the standard model rejects the project.

In equation (8), the parameter η reflects only aversion to intertemporal fluctuations. Therefore, the term $\eta^2 \frac{\sigma^2}{2}$ is interpreted as the cost of expected fluctuations triggered by the aversion to non-smooth intertemporal consumption paths. I refer to the expression as “the standard risk term”, as it is the only expression capturing risk in an analysis based on the standard model. In the case of fully disentangled preferences (scenario D, see section 2.3), the magnitude of the intertemporal risk aversion contribution is

$$\frac{RIRA |1 - \eta^2| \frac{\sigma^2}{2}}{\eta^2 \frac{\sigma^2}{2}} \approx 33,$$

times that of the standard risk contribution. Figure 1 depicts the different discounting contributions as a function of η . The graph sets $RRA = 9.5$, $\mu_t = 2\%$ and $\sigma_t = 4\%$.

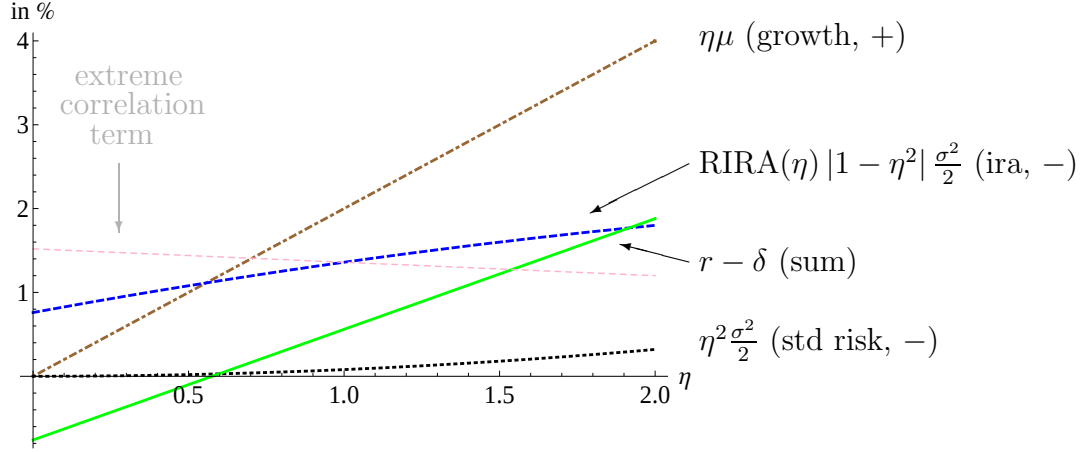


Figure 1 depicts the different contributions of the discount rate as well as the total discount rate net of pure time preference $r - \delta$. The terms are drawn as a function of aversion to intertemporal fluctuations η , keeping relative Arrow-Pratt risk aversion fixed at $RRA = 9.5$ and using $\mu_t = 2\%$ and $\sigma_t = 4\%$. A minus sign in the bracket implies that the term is negative and subtracted from the positive growth term. The abbreviation *ira* denotes the contribution from intertemporal risk aversion.

The positive growth term (brown, dash-dotted) dominates for reasonably high values of η . The intertemporal risk aversion term (blue, dashed) defines the main reduction. The standard risk term (black, dotted) plays a very minor role in determining the overall discount rate net of pure time preference (green, solid). Note that the intertemporal risk aversion contribution is continuous at $\eta = 1$. Keeping RRA fixed, $RIRA$ is itself a function of η as it measures the difference between Arrow-Pratt risk aversion and aversion to intertemporal substitution.²⁰ Moving from the standard model with $RRA = \eta = 2$ to the disentangled model with $\eta = \frac{2}{3}$ implies two changes in the discount rate. First, the growth effect is significantly reduced once η captures only the attitude to intertemporal substitution (brown dash-dotted line is evaluated further to the left). This first effect is an *indirect* of disentanglement. It merely corrects the estimate of the growth term in the standard Ramsey equation. Second, intertemporal risk aversion reduces the discount rate (blue dashed line is now subtracted from the brown dashed line). This second effect is the *direct* effect of intertemporal risk aversion. When moving in Figure 1 from $\eta = 2$ all the way to $\eta = \frac{2}{3}$, the indirect effect implies an even larger reduction in the discount

²⁰The Epstein-Zin preference representation in equation (5) implies a switch in the sign of utility when η crosses unity. During this sign switch $1 - \eta$ goes through zero, while $RIRA$ has a pole. One could redefine $RIRA |1 - \eta^2|$ as the actual measure of intertemporal risk aversion, as it is positive if and only if equation (4) holds. I stick to the definition in equation (7) because this measure is completely analogous to the measure suggested for smooth ambiguity aversion.

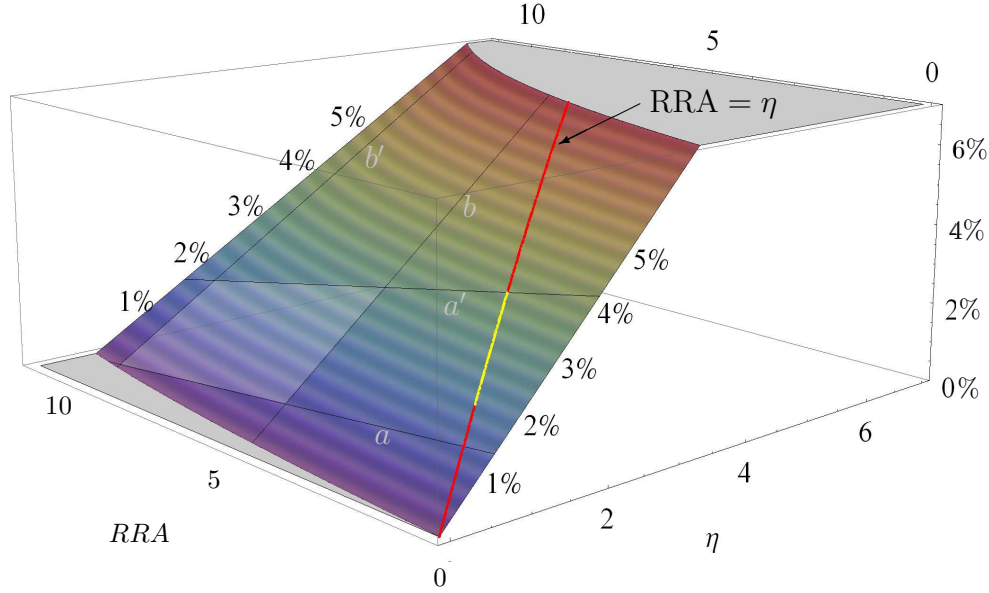


Figure 2 depicts the total discount rate net of pure time preference $r - \delta$ as a function of η and RRA. Moving along the thick (red and yellow) line keeps $RRA = \eta$, representing the only movement possible in the standard model. The yellow part of the line marks the most common parameterization $\eta \in [1, 2]$. Increasing risk aversion along this line necessarily leads to high discount rates. In contrast, increasing risk aversion in the disentangled model corresponds to a movement to the left and reduces the discount rate. The two thin black lines a and a' from right to left hold η fix at $\frac{2}{3}$ (as in D) and 2 (as in N), while increasing risk aversion. The two thin lines b and b' moving up hold RRA fix at 5 and 10, while increasing η . The cited estimates of the disentangled model all imply rates in the lowest left corner of the shaded area between the thin black lines. It is $\mu_t = 2\%$ and $\sigma_t = 4\%$.

rate than the direct effect.

Figure 2 illustrates the crucial difference between the standard model and a model of general risk attitude in terms of the resulting discount rates. The standard model confines $RRA = \eta$. The thick colored line moving upwards from the origin depicts the discount rate net of pure time preference $r - \delta$ in the standard model. The yellow region of the otherwise red line reflects the most common preference specifications, $\eta \in [1, 2]$. Accounting for higher risk aversion in the standard model moves $r - \delta$ up along the $RRA = \eta$ line and significantly increases the discount rate. In contrast, higher risk aversion in the disentangled model decreases the discount rate. The thin black lines going from right to left increase risk aversion while keeping $\eta = \frac{2}{3}$ (D) and $\eta = 2$ (N). The thin black lines moving up increase η while fixing RRA at 5 and 10. All of the cited estimates of the disentangled model imply discount rates in the lowest corner of the shaded area between these lines. In contrast, an attempt to accommodate observed risk aversion of $RRA \in [5, 10]$ in the standard model would imply discount rates far

above the 7% bound of the graph (plus pure time preference).

Gollier (2002) derives a social discount rate that closely relates to the rate characterized in equation (8). It differs in two respects. First, only equation (8) decomposes the discount rate into terms that resemble the rate in the standard model, and a term capturing the deviation from the standard model. The term capturing this deviation, which is proportional to intertemporal risk aversion, fleshes out the magnitude of the difference between employing the standard model and the comprehensive model. Moreover, the term will be crucial in drawing the parallel to discounting under smooth ambiguity aversion. Second, his derivation of the isoelastic special case of his model is an approximation and assumes that the support of the growth rate is a small neighborhood around zero. In contrast, the normal growth distribution assumed here has infinite support. Contrary to a seemingly wide-spread belief, Gollier’s equation neither holds for normal growth, nor does he claim so. For normal growth, equation (14) in Gollier (2002) misses a term $\eta \frac{\sigma^2}{2}$. Gollier’s derivation starts with the Arrow-Pratt approximation of the certainty equivalent. The Arrow-Pratt approximation is globally exact for normally distributed risk and CARA utility, i.e., for utility or risk aversion functions that capture constant *absolute* aversion rather than constant *relative* aversion as assumed by isoelastic preferences. For short time horizons, low growth, and low variance, Gollier’s approximation is good in the case of a normal growth distribution and CARA aggregators. I discuss these results in Appendix C.

3.2 Stochastic Projects

In the preceding section, uncertainty is about economic growth. Many long-term investment projects, however, are characterized by uncertain payoffs. Greenhouse gas mitigation and climate change adaptation projects are important example. Once stochasticity of the project is introduced, the correlation between project payoff and uncertain economic growth becomes crucial for valuation. Lind (1982) argues for full positive correlation between project payoffs and economic baseline growth. Weitzman (2007) points out that this standard approach to cost benefit analysis does not apply to climate change projects. The major areas impacted by climate change would be “ ‘outdoor’ aspects (broadly defined) like agriculture, coastal recreational areas, and natural landscapes”, which are little correlated to technological progress. Moreover, some of these impacts directly affect utility rather than production. Various economists used related arguments to promote the use of the risk-free rate for the assessment of climate change projects. Indeed, the risk-free rate coincides with the discount rate for an uncorrelated stochastic

project (see below).

Some common integrated assessment models imply a different correlation mechanism. Nordhaus (2008) notes that a high growth realization implies more production and, thus, more emissions. Then, damages and abatement payoff are both high. The resulting positive correlation between growth (or wealth) and project payoff is driven by the production-emissions-damage link. I add a third consideration driving correlation in the climate change context. The causal chain pointed out by Nordhaus (2008) and captured in his integrated assessment model DICE relies on the exogenous growth rate affecting emissions. However, if climate change turns out to have severe economic impact, then it is likely to affect the economic growth rate (Pindyck 2011). A possible transmission channel from climate change to economic growth is the mere straining of resources that would otherwise lead to technological progress. Another channel could divert general technological progress into adaptation technology that merely serves to maintain the status quo. Dell, Jones & Olken (2008) find evidence that a similar channel indeed affects the growth rates in developing economies even at the moderate levels of climate change experienced in the past. A further transmission channel can be a distributional tension caused by fresh water scarcity, droughts, and agricultural impacts in some regions of the world, triggering social conflict within a society as well as international conflicts. In general, all three correlation arguments (Weitman's, Nordhaus', and the one added here) apply to the evaluation of climate change related projects and the integrated assessment of climate change under uncertainty.

This section derives the discounting formula for projects that are correlated with economic baseline growth. I show that general risk attitude creates a much more important role for correlation than in the standard model. For stochastic projects, the decision maker no longer trades a deterministic unit of consumption between the present and the future. Formally, she trades a marginal unit dx_1 of her current certain consumption x_1 against a marginal fraction $d\epsilon$ of a stochastic project y with expected unit payoff, i.e. $Ey = 1$. The future payoff y is correlated with uncertain future baseline consumption x_2 . The stochastic discount rate is characterized as $r = \ln \frac{d\epsilon}{-dx_1}$ for an intertemporal trade-off that leaves overall welfare constant:

$$0 = \frac{d}{dx_1} u(x_1) dx_1 + \beta \frac{d}{d\epsilon} f^{-1} [E_{p(x_2, y)} f \circ u(x_2 + \epsilon y)] \Big|_{\epsilon=0} d\epsilon. \quad (9)$$

I briefly comment on this extension of the social discounting model. First, for a certain project the marginal payoff ϵy is certain and corresponds to dx_2 in the usual derivation of the risk-free social discount rate.²¹ Second, marginality in the trade-off that defines the

²¹In this case the formula above reduces to a more precise notation of what is commonly written as

discount rate plays the same role as in any other economic price concept. The analytic formula for the discount rate will characterize (in rates) the present value willingness to pay for a marginal unit of the stochastic project. This willingness to pay depends on correlation. Third, I formalize a trade-off between a marginal current unit and the first marginal part of a finite stochastic unit project y .²² Fourth, observe that the derivation does not rely on an optimal allocation of an adaptation-mitigation-portfolio. Such an optimality assumption would be inadequate in the climate change application that I discuss.

I assume that the uncertain productivity of the project $\ln y$ and the stochastic economic consumption growth rate g are jointly normally distributed with standard deviations σ_y , σ_g , and correlation κ . The expected growth rate is denoted μ_g and the condition $Ey = 1$ determines the remaining parameter of the distribution.²³

Proposition 2: The stochastic social discount rate in the isoelastic setting with intertemporal risk aversion is

$$r = \delta + \eta\mu_g - \eta^2 \frac{\sigma_g^2}{2} - \text{RIRA} \left| 1 - \eta^2 \right| \frac{\sigma_g^2}{2} + \eta\kappa \sigma_g \sigma_y + |1 - \eta| \text{RIRA} \kappa \sigma_g \sigma_y. \quad (10)$$

The second line distinguishes the stochastic social discount rate from its risk-free relative of the previous section. In the case of certainty about the project, this second line vanishes ($\sigma_y = 0$). The same is true if the risk of the project and the baseline scenario are uncorrelated ($\kappa = 0$). The discount rate characterizes a marginal shift between current consumption and uncertain future consumption. Therefore, risk aversion with respect to the marginal project itself is a second order effect that does not find its way into the discount rate. Stochasticity of the small project only contributes through its interaction with baseline uncertainty. The second term in the second line of equation

$\frac{d}{dx_2} \dots E_p \dots u(x_2)$: the above notation makes explicit that (for $y = 1$) the marginal unit (ϵ or dx_2) in the decision maker's trade-off is certain, while the baseline x_2 is uncertain. Observe that also the first period derivative in equation (9) can be rewritten as $\frac{d}{d\epsilon_1} u(x_1 + \epsilon_1 y_1) |_{\epsilon_1=0, y_1=1} d\epsilon_1$.

²²Modeling an infinitesimal share of a non-marginal unit project rather than a marginal project itself is important. It is well known that risk effects are second order effects. Therefore, stochasticity effects of an infinitesimal project would vanish.

²³Let μ_y denote the expected value of (the marginal distribution of) $\ln y$. The condition $Ey = 1$ implies $\mu_y = -\frac{\sigma_y^2}{2}$. Making use of this constraint, it is $\text{Var}(y) = e^{\sigma_y^2} - 1 \approx \sigma_y^2 + \frac{\sigma_y^4}{2}$. Thus, in the percentage range, σ_y also approximates well the standard deviation of the project y itself. I refer to κ as the correlation between the project and the baseline even though, more precisely, it is the correlation between $\ln y$ and the growth rate $g = \ln \frac{x_2}{x_1}$.

(10) distinguishes the correlation contribution in a model including intertemporal risk aversion from the correlation contribution in the standard model.

I assess the magnitude of the correlation contribution in the same growth scenario as before with $\mu = 2\%$ and $\sigma_y = \sigma_g = 4\%$. I discuss and relax the assumption that the standard deviation of the project payoff equals that of the growth process in the next section. The correlation multiplier in scenario N, the standard model with Nordhaus' preferences, is $\eta\sigma_g\sigma_y = 0.3\%$. In contrast, scenario D (disentangling $\eta = \frac{2}{3}$ from $\text{RRA} = 9.5$) reduces the standard multiplier of the correlation coefficient to $\eta\sigma_g\sigma_y = 0.1\%$, but adds an intertemporal risk aversion multiplier of $|1 - \eta| \text{RRA} \sigma_g\sigma_y = 1.4\%$. The correlation contributions to the social discount rate are proportional to these multipliers and the correlation coefficient. For example, a correlation of $\kappa = \pm 0.5$ increases the social discount rate in the disentangled scenario by ± 0.8 to an overall rate of 2.4% and 0.9%, respectively (for $\delta = 1.5\%$). Thus, under intertemporal risk aversion the correlation between the project payoff and economic growth is of first order importance for the discount rate. Figure 1 shows the dominant correlation multiplier caused by intertemporal risk aversion as a function of η in light gray. It is the amount added (subtracted) from the risk free rate when accounting for full positive (negative) correlation and comprehensive risk attitude.

3.3 The Relevance of Future Risk

How relevant is uncertainty for the evaluation of long-term projects? Weitzman (2009) emphasizes the importance of uncertainty about climate sensitivity and economic damages for the assessment of climate change policies. His analysis builds crucially on generating fat tails in a standard expected utility model. His interesting findings have been criticized in a series of papers for their assumptions about the climate system as well as for stretching a too simple economic trade-off model beyond the domain where it is meaningful (Horowitz & Lange 2009, Pindyck 2009, Nordhaus 2009, Millner 2011). The current paper opens up a very different perspective on how uncertainty affects climate change evaluation. Even without uncertainty about the climate system itself, uncertainty about economic growth has a major impact on optimal climate policy. Including uncertainty about the climate system, the interaction and correlation between growth and project payoffs becomes a major ingredient for evaluating climate change and pricing carbon. The section analyzes the relevance of growth and project uncertainty in the model of comprehensive risk attitude.

The previous sections have shown that growth uncertainty reduces the discount rate.

The two period model of those sections is equivalent to a simple iid growth model. However, once uncertainty becomes persistent, it is well known that uncertainty not only changes the level of the discount rate, but also its term structure (Weitzman 1998, Azfar 1999). Making the time step explicit in equation (8) results in

$$r_T = \delta T + \eta\mu_T - \eta^2 \frac{\sigma_T^2}{2} - \text{RIRA} \left| 1 - \eta^2 \right| \frac{\sigma_T^2}{2},$$

where variables indexed by T depend on the time horizon (payoff period). For an iid process like a Brownian motion the variance grows linearly in futurity T . Then, if expected growth is constant ($\mu_T = T\mu$), payoffs in period T are simply discounted at T times the constant rate stated in equation (8). However, with persistent uncertainty, the variance grows faster and the term structure of the discount rate falls: payoffs in the distant future are discounted at a relatively lower (yearly average) discount rate than payoffs in the close future.

In the following, I analyze the importance of uncertainty for the evaluation of climate change, comparing the models with and without a comprehensive representation of risk attitude. The analysis relies on the primordial importance of the discount rate for climate change evaluation, impressively documented in Nordhaus's (2007) simulation discussed in the introduction. I build the analysis around the following question: At what level of riskiness do uncertainty effects cancel the growth effect in the social discount rate? Growth discounting is the main economic driver of discounting. If uncertainty effects cancel the growth effect, then future costs and benefits are solely discounted with the pure rate of time preference δ . I compare the necessary risk level between the standard and the disentangled model, and I analyze how this risk depends on the correlation between growth and project payoffs. A major advantage of approaching the uncertainty comparison in this way is that the uncertainty analysis is independent of pure time preference – a parameter whose magnitude is most contested in the debate.

The analysis uses a time horizon (or period) of 50 years. I now discuss long-term growth uncertainty, a highly intertemporally correlated event, instead of discussing a representative year in an iid growth scenario. I keep the assumption of an expected 2% yearly growth rate of consumption. In 50 years, climate change is going to affect our planet severely under almost any forecast. It will affect economic activities directly as well as non-produced consumption. Some events like changes in precipitation patterns (or land loss) can also cause social unrest or war.²⁴ I will measure uncertainty in terms

²⁴Similarly, learning about climate sensitivity and, thus, an important ingredient for damages, takes place on a similar time scale (Kelly & Kolstad 2001, Keller, Bolkerand & Bradford 2004).

of the variance of the growth process (and the project payoff) and translate it into the probability of being worse off tomorrow than today.

Corollary 1: The discount rate reduces to pure time preference, i.e. $r_T = \delta T$, if and only if,

1. in the case of the risk-free rate

$$\sigma_T = \left(\frac{1}{2} \left(\eta + \frac{|1 - \eta^2|}{\eta} \text{RIRA} \right) \right)^{-\frac{1}{2}} \mu_T^{\frac{1}{2}}. \quad (11)$$

2. in the case of a risky project with $\sigma_T = \sigma_{gT} = \sigma_{yT}$

$$\sigma_T = \left(\frac{1}{2} \left(\eta + \frac{|1 - \eta^2|}{\eta} \text{RIRA} \right) - \kappa \left(1 + \frac{|1 - \eta|}{\eta} \text{RIRA} \right) \right)^{-\frac{1}{2}} \mu_T^{\frac{1}{2}}. \quad (12)$$

3. in the case of a general risky project

$$\sigma_{yT} = \frac{(\eta^2 + |1 - \eta^2| \text{RIRA}) \sigma_{gT} - 2\eta \frac{\mu_T}{\sigma_{gT}}}{2\kappa (\eta + |1 - \eta| \text{RIRA})}.$$

The conditions for eliminating the growth effect are identical for the risk-free rate and for the case of a risky project whose payoffs are uncorrelated to overall growth. More uncertainty is required, if the risk terms are to cancel the growth term, for a project whose payoffs are positively correlated to growth uncertainty. If the expected growth rate is simply $\mu_T = \mu T$, with a constant yearly expectation of μ , then equations (11) and (12) show that (only) a standard deviation that evolves proportional to $\sqrt{T}$ leaves the yearly discount rate constant (at pure time preference). This fact illustrates once again that the term structure of the discount rate is flat only for iid uncertainty where $\sigma_T \propto \sqrt{T}$.

I analyze Corollary 1 using concrete probabilistic events. By $p^* \equiv P(x_{50} \leq x_1)$ I denote the probability that anything including climate change causes society to be worse off in $T = 50$ years than today. It is the probability mass in the thin left tail of the growth distribution that implies a non-increasing standard of living between today and in 50 years. For the subsequent simulations, I keep expected consumption growth at a yearly rate of 2% and $T = 50$, which implies $\mu_T = 1$. Table 1 summarizes the numerical results for the different preference representations and for differing degrees of correlation. The table follows part 2 of Corollary 1 assuming $\sigma_{gT=50} = \sigma_{yT=50}$ (relaxed further below). In scenario N, the intertemporally expected utility standard model, a standard deviation

The Social Discount Rate under Intertemporal Risk Aversion and Ambiguity

Table 1 determines the risk that reduces the social discount rate to pure time preference.

	η	RRA	RIRA	$\kappa = -1$		$\kappa = -0.5$		$\kappa = 0$		$\kappa = 0.5$		$\kappa = 1$	
				σ %	p^* %	σ %	p^* %	σ %	p^* %	σ %	p^* %	σ %	p^* %
N	2	2	0	71	7.9	82	11	100	16	140	24	/	/
D	$\frac{2}{3}$	9.5	27	20	0.00002	23	0.0009	30	0.04	49	2	/	/
S1	2	9.5	7.5	30	0.04	33	0.13	39	0.5	49	2	73	8.5
S2	2	5	3	42	0.8	47	1.7	55	3.6	71	7.9	120	19

Notes: $\sigma = \sigma_y = \sigma_g$ =standard deviation; p^* =probability of being worse off in 50 years than today; κ =correlation coefficient between project and baseline risk. The $\kappa = 0$ case is equivalent to the risk-free social discount rate. Expected growth is a yearly 2% over 50 years. The settings are ‘N’ based on Nordhaus, ‘D’ for the disentangled parameter estimates (section 2.3), and sensitivity scenarios ‘S1’ and ‘S2’.

of unity is necessary to eliminates growth discounting from the risk-free discount rate ($\kappa = 0$). This standard deviation translates into the large probability of $p^* = 16\%$ that society is equal or worse off in 50 years. In contrast, the disentangled scenario D with a comprehensive treatment of risk attitude implies $\sigma = 0.3$ and $p^* = 0.04\%$. A chance of 4 in 10 000 that we might not be better off in 50 years than today seems quite reasonable. Then, we should not discount the future for growth in the disentangled model. The probability necessary in the standard model is 400 times larger. The sensitivity scenario S1 in the table leaves relative Arrow Pratt risk aversion RRA at the estimate of 9.5, but increases aversion to intertemporal substitution η to Nordhaus’ value of 2. This change reduces intertemporal risk aversion, but still results in a probability p^* necessary to reduce discounting to pure time preference that is only a 30st of the probability needed in scenario N. Sensitivity scenario S2 further reduces intertemporal risk aversion by also lowering the Arrow Pratt coefficient of risk aversion to 5. Then, p^* is about a fifth of the corresponding probability in scenario N.

The table also shows the important role played by the correlation between project payoff and growth uncertainty in the disentangled approach: correlation can change the probability p^* by several orders of magnitude. With disentangled preferences and a correlation coefficient $\kappa = -.5$, a probability of $p^* = 0.0009\%$ is sufficient to make the risk terms cancel the growth effect, yielding a social discount rate that is equivalent to pure time preference. In contrast, with a correlation coefficient $\kappa = +.5$, a probability of $p^* = 2\%$ would be needed. Under standard preferences these probabilities would be 11% and 24%, respectively. The stochasticity of the project with expected unit payoff can be characterized as follows. Let $p_y = P(y < 0.5 \vee y > 2)$ denote the probability that the project pays less than half or more than double of the expected unit. The

interval $\sigma_y \in [0.2, 0.3]$ found for a non-positive correlation in the disentangled approach translates into $p_y \in [0.1\%, 2.2\%]$, whereas the corresponding interval $\sigma_y \in [.7, 1]$ in the N scenario translates into $p_y \in [35\%, 54\%]$. For perfect positive correlation $\kappa = 1$ the risk effects can only cancel the growth effect if the standard deviation of baseline growth exceeds that of the stochastic project. Thus, condition (12) has no solution.²⁵

Disentangling the two different risks yields further insight. The left graph in Figure 3 depicts combinations of standard deviations that reduce the social discount rate to pure time preference. The right graph translates these standard deviations into the probabilities p^* that society is equal or worse off in 50 years under the expected yearly growth rate of 2% (growth uncertainty) and into the probability p_y that the project pays out less than half or more than double the expected unit.²⁶ The dashed lines correspond to disentangled preferences (D), while the solid lines correspond to Nordhaus preferences (N). The graphs demonstrate that more uncertainty of the stochastic project decreases the baseline risk necessary for a reduction of the discount rate, if and only if, the correlation is negative. For a positive correlation, higher project uncertainty also requires a higher volatility of baseline growth if risk effects are to cancel the growth effect. The graphs clearly show the importance of the correlation coefficient already for low levels of the project's payoff uncertainty p_y . Moreover, the graphs reiterate the order of magnitude difference resulting from entangled versus disentangled preferences.

4 Ambiguity Aversion and Second Order Uncertainty

4.1 Ambiguity

A different shortcoming of the intertemporally additive expected utility standard model is its assumption that uncertainty can be described by a unique probability measure. In many real world applications, these probability distributions or risks are unknown. Different strands of literature capture non-risk uncertainty under the names deep uncertainty, hard uncertainty, or ambiguity. In this paper, I employ and extend Klibanoff et al.'s (2005) model of smooth ambiguity aversion (KMM) and show how ambiguity affects the discount rate. In contrast to many models of ambiguity and deep uncer-

²⁵The entries in Table 1 correspond to the intersections of the corresponding curves on the left of Figure 3 with the dotted 45° line. The shape of the curves for $\kappa = 1$ demonstrates why there is no solution to equation (12) (no intersection of the $\kappa = 1$ curves with the dotted line).

²⁶Note that such a translation into probabilities is possible because the marginal distribution of the bivariate normal only depends on the volatility in the remaining dimension. Also note that the vertical range of the right graph corresponds to a σ_y -range of $[0, 0.5]$.

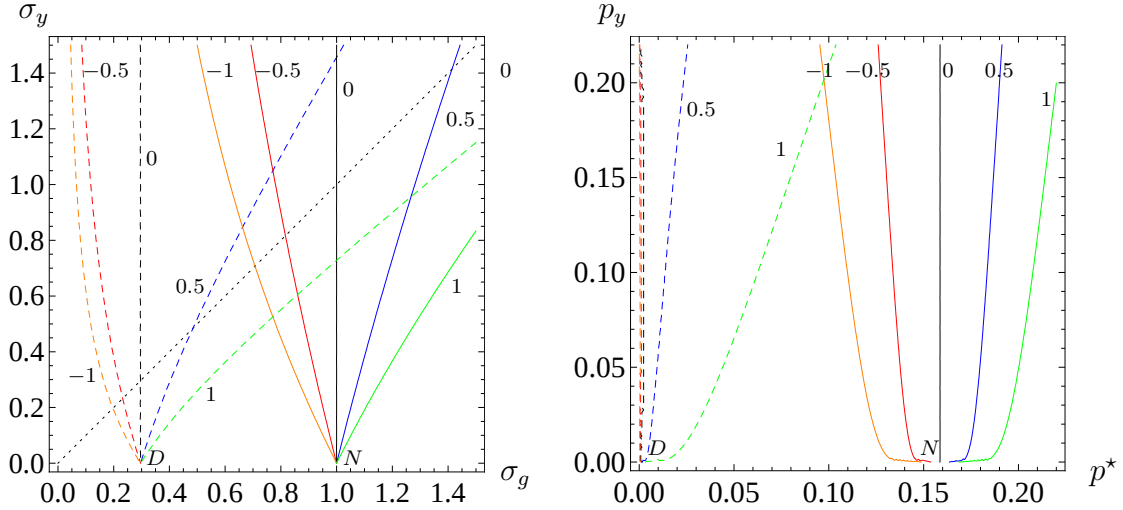


Figure 3 depicts the combinations of standard deviations (left) and probabilities (right) of baseline growth (horizontal axis) and project payoff (vertical axis) implying a discount rate reduction to the rate of pure time preference. p^* represents the probability of being worse off in 50 years than today under a normally distributed growth rate with expected value of 2% per year. p_y represents the probability that the payoff of the stochastic unit project lies outside of the interval $[0.5, 2]$. The numbers labeling the curves denote the correlation κ between baseline growth and project payoff. The dashed curves (originating at 'D') are based on the disentangled approach, the solid curves (originating at 'N') are based on Nordhaus' entangled preferences. The intersections of the curves in the left graph with the dotted line (identity) depicts the σ values reported in Table 1.

tainty, the KMM model satisfies normative desiderata including time consistency and other rationality constraints. I show the close similarity of this ambiguity model to the model of intertemporal risk aversion. The KMM model captures uncertainty about the correct objective probability distribution in terms of second order uncertainty: a subjective probability distribution over objective probability distributions or risk. The model is particularly interesting and applicable in the context of climate change and long-term economic growth: in both situations we face too little data and insufficient knowledge about the underlying model to derive confident probabilistic estimates governing the future.

The basic structure of the model is similar to a Bayesian prior model. The Bayesian prior is interpreted as ambiguous second order uncertainty. The crucial distinction between the smooth ambiguity and the standard Bayesian model lies once more in the preference representation that accompanies the uncertainty model. In a standard Bayesian expected utility model, a decision maker evaluates objective first order probabilities and

subjective second order probabilities with the same risk aversion function (simultaneously capturing aversion to intertemporal substitution). In contrast, the KMM model incorporates the finding that individuals generally prefer objective risk to subjective risk. For this purpose, the model introduces a new measure of risk aversion for ambiguous lotteries, i.e., the subjective second order probability distributions. I will explain that this measure of ambiguity is a close analogue to the measure of intertemporal risk aversion. The original KMM model does not disentangle attitudes in the case of objective risk. By introducing ambiguity aversion, the model introduces intertemporal risk aversion only to subjective lotteries, while keeping intertemporal risk neutrality for objective lotteries. I extend the model to capture both, disentangled aversion to subjective and to objective risk. I show that the resulting social discounting model is a clone of the model discussed in the previous sections.

The decision-theoretic literature has developed a range of different approaches to capture situations of ambiguity. I briefly survey the most important ones in the remainder of this section. One way to characterize non-risk uncertainty is by extending the concept of probabilities to more general set functions called “capacities”. These set functions weigh possible events but are not necessarily additive in the union of disjoint events. Because of this non-additivity, the standard measure integral has to be exchanged for the more general Choquet integral in order to calculate expected utility, giving rise to the name “Choquet expected utility”. A second approach defines an evaluation functional that expresses beliefs in the form of sets of probability distributions rather than unique probability distributions. The first and simplest such representation goes back to Gilboa & Schmeidler (1989). Here a decision maker evaluates a scenario by taking expected values with respect to every probability distribution deemed possible and then identifies the scenario with the minimal expected value in this set.²⁷ A more general representation of this type is given by Ghirardato, Maccheroni & Marinacci (2004), Maccheroni, Marinacci & Rustichini (2006*a*), and, in an intertemporal framework, Maccheroni, Marinacci & Rustichini (2006*b*). There are several equivalence results between the Choquet approach and that of multiple priors as well as rank dependent utility theory where a decision maker uses distorted probabilities in an expected utility approach increasing the weights given to small probability events. In the climate change context, the main advantage of the smooth ambiguity model over these alternatives is its normative attractiveness achieved by maintaining time consistency and the essence of the independence

²⁷Hansen & Sargent (2001) give conditions under which this approach is equivalent to what is known as robust control or model uncertainty, which again has overlapping representations with the model of constant absolute intertemporal risk aversion presented in Traeger (2007).

axiom. Just as importantly for my purposes, I want to show that the KMM model is closely related to the model of intertemporal risk aversion and yields similar discounting results. Finally, its similarity to the Bayesian framework makes the model not only easy to interpret, but also allows me to relate to Weitzman's (2009) discourse on structural uncertainty.

4.2 The Generalized Model of Smooth Ambiguity Aversion

The section introduces the smooth ambiguity aversion model by Klibanoff et al. (2005) and, in the intertemporal setting, by Klibanoff et al. (2009). It represents ambiguity (non-risk uncertainty) as second order probability distributions, i.e. probabilities over probabilities. The model introduces a different attitude for evaluating second order uncertainty as compared to first order risk. Translated into the simple setting of this paper, the generally recursive evaluation of the future writes as

$$V(x_1, p, \pi) = u(x_1) + \beta \Phi^{-1} \left\{ \int_{\Theta} \Phi [E_{p_{\theta}(x_2)} u(x_2)] d\pi(\theta) \right\} .$$

For a given parameter θ , the probability measure p_{θ} on the consumption space X denotes first order or "objective" probabilities over consumption. The expectation operator takes expectations with respect to p_{θ} . However, the parameter θ and, thus, the correct objective probability distribution are unknown. The probability measure π denotes the prior over the parameter $\theta \in \Theta$.²⁸

In Klibanoff et al.'s setting, the utility function u corresponds to the utility function of the standard model. It jointly captures aversion to intertemporal substitutability and "objective" or first order risk. The function Φ captures additional aversion with respect to second order uncertainty, which they call *ambiguity aversion*. For Φ linear, the model collapses to the standard Bayesian model. The coefficient describing relative ambiguity aversion is defined as

$$\text{RAA} = \frac{\Phi''(z)}{\Phi'(z)} |z| .$$

A different perspective on ambiguity aversion in the KMM model is as follows. The utility function captures aversion to intertemporal substitution. Aversion to objective risk is only captured to the degree that risk generates intertemporal fluctuations in

²⁸I adopt a continuous parameter space Θ , whereas this parameter space is finite in Klibanoff et al.'s (2009) axiomatization of the model. Moreover, Klibanoff et al. (2005, 2009) setting features acts rather than probability measures on the outcome space.

the consumption path. In contrast, the evaluation of the subjective prior incorporates intrinsic aversion to ambiguity, just as intertemporal risk aversion incorporated intrinsic aversion to risk. I noted that intrinsic aversion is, e.g., able to explain asset pricing puzzles. However, the current model only picks up intrinsic aversion to ambiguity, not to risk. Thus, it remains unclear what part of the ambiguity aversion effects are driven by additional aversion to subjective as opposed to objective risk, and what effects that are labeled ambiguity aversion are merely driven by uncertainty aversion dominating the propensity to smooth consumption over time.

In this paper, I combine Klibanoff et al.'s model of ambiguity aversion with my model of intertemporal risk aversion, leading to a welfare representation of the form

$$V(x_1, p, \pi) = u(x_1) + \beta \Phi^{-1} \left\{ \int_{\Theta} \Phi [f^{-1} E_{p_{\theta}(x_2)} f \circ u(x_2)] d\pi(\theta) \right\}. \quad (13)$$

In this generalization, u characterizes aversion to intertemporal substitution only, f characterizes intertemporal risk aversion, and Φ characterizes ambiguity aversion.²⁹ In the representation of equation (13), ambiguity aversion characterizes attitude with respect to second order uncertainty similar to the way that intertemporal risk aversion characterizes attitude with respect to first order risk. This parallel is a fundamental insight about the smooth ambiguity model and will also emerge in the expression for the discount rate. The current generalized framework permits a threefold disentanglement of risk aversion, ambiguity aversion, and aversion to intertemporal substitution. To enable an analytic derivation of the social discount rate, I will once more revert to the isoelastic setting. In addition to the earlier assumptions of section 2.2 and equation (6), I assume $\Phi(z) = (\rho z)^{\varphi}$, which yields a coefficient of relative ambiguity aversion

$$\text{RAA} = \begin{cases} 1 - \varphi & \text{if } \rho > 0 \\ \varphi - 1 & \text{if } \rho < 0. \end{cases}$$

4.3 The Social Discount Rate and Ambiguity about Growth

In the context of climate change, Weitzman (2009) recently argued that the parameters of the distribution governing economic growth are unknown. Like Weitzman, I adopt a Bayesian setting to capture such second order uncertainty. However, Weitzman sticks with the standard risk model underlying equation (2), in contrast, I introduce ambiguity

²⁹In an alternative representation, I could apply the inverse of the function f characterizing intertemporal risk aversion in front of Φ^{-1} instead of its current position where it acts on the expected value operator. Then, the same preferences are represented with a different function Φ that would characterize only “access aversion” to ambiguity as opposed intertemporal risk aversion.

attitude as formulated by Klibanoff et al. (2005, 2009) as well as intertemporal risk aversion. Taking the simplest example of Bayesian second order uncertainty, I assume that *expected* growth is itself a normally distributed parameter θ with expectation μ and variance τ^2 . Formally, first order uncertainty p_θ is once more a joint normal distribution over economic growth and project payoff (see section 3.2). However, the expected growth rate is unknown, and I assume $E(g|\theta) \sim N(\theta, \sigma^2)$ and $\theta \sim N(\mu, \tau^2)$, preserving the interpretation of μ as the overall expectation of the growth trend. The special case of Proposition 3 for RIRA = 0 and $\kappa = 0$ has independently been derived by Gierlinger & Gollier (2008).

Proposition 3: The stochastic social discount rate in the isoelastic setting with intertemporal risk aversion and ambiguity about expected growth is

$$\begin{aligned} r = \delta + \eta \mu_g - \eta^2 \frac{\sigma_g^2 + \tau^2}{2} - \text{RIRA} \left| 1 - \eta^2 \right| \frac{\sigma_g^2}{2} \\ + \eta \kappa \sigma_g \sigma_y + |1 - \eta| \text{RIRA} \kappa \sigma_g \sigma_y \\ - \text{RAA} \left| 1 - \eta^2 \right| \frac{\tau^2}{2}. \end{aligned} \quad (14)$$

The first two terms on the right hand side reflect, once more, the discount rate in the standard Ramsey equation under certainty. The third term $-\eta^2 \frac{\sigma_g^2 + \tau^2}{2}$ reflects the well-known extension for risk. Note that the overall variance of the growth process is now $\sigma^2 + \tau^2$ because of the additional layer of uncertainty characterized by the second order variance τ^2 . The second line gives the corrections if the project is stochastic. This correction remains as in the previous section. The third line characterizes the new contribution to intertemporal value development due to ambiguity aversion. The term is proportional to second order variance τ^2 , relative ambiguity aversion RAA, and the term $|1 - \eta^2|$ already encountered in the correction of the social discount rate for intertemporal risk aversion. In fact, the contribution of ambiguity aversion is formally equivalent to the contribution of intertemporal risk aversion, replacing first by second order variance and RIRA by RAA. Proposition 3 provides a full disentanglement between the contributions already present under certainty, those arising under risk only reflecting aversion to intertemporal fluctuations, the contributions driven by intrinsic risk aversion, and those brought about by aversion to ambiguity.

Quantitatively, a decision maker who is more averse to ambiguity than to risk will lower the discount rate more for second order variance (ambiguity) than for first order variance (risk). Otherwise, the discussion from section 3 stays qualitatively the same. In general, an ambiguity averse decision maker will employ a lower (risk-free or stochastic)

discount rate when the baseline scenario is ambiguous. He is willing to invest in a certain or stochastic project with relatively lower productivity than is a decision maker who is ambiguity neutral or just faces (first order) risk.

Relating my result to Weitzman (2009), I ignore everything but the first three terms on the right of equation (14). The only difference between these remaining terms and the standard equation (2) is the additional variance τ^2 in the third term (standard risk term). This additional variance is a straightforward consequence of making the growth process more uncertain by introducing a prior over some parameter of the growth process. In the case of the normal distributions adopted here, the variance simply adds up. From the given example, it is difficult to see how adding a Bayesian prior would bring the standard risk term back into the order of magnitude comparable to the other terms of the social discount rate. Instead of a doubling, a factor of 10 – 100 is needed. The only way to reach this result is by sufficiently increasing the variance of the prior. Effectively, this is what Weitzman (2009) does in deriving what he calls a dismal theorem. He introduces a fat tailed (improper) prior whose moments do not exist. Consequently, the risk-free social discount rate in equation (14) goes to minus infinity implying an infinite willingness to transfer (the first unit of certain) consumption into the future. Weitzman limits this willingness by the value of a (or society's) statistical life.³⁰ Instead of augmenting uncertainty, the above proposition introduces ambiguity aversion, i.e. the term $\text{RAA} |1 - \eta^2| \frac{\tau^2}{2}$, into social discounting, reflecting experimental evidence that economic agents tend to be more afraid of unknown probabilities than they are of known probabilities (most famously, Ellsberg 1961).

Observe that intertemporal risk aversion affects the correlation terms, while ambiguity aversion does not. Uncertainty aversion enters the stage if the corresponding uncertainty affects overall welfare. If uncertainty only affects the payoff of the marginal project, then aversion is a second order effect absent from the social discount rate. Thus, growth variance σ_g^2 directly interacts with intertemporal risk aversion, and uncertainty over the expected growth rate, captured by τ^2 , directly interacts with ambiguity aversion. Moreover, correlation between overall economic growth and the project payoff interacts growth uncertainty and the marginal project payoff. This interaction results in the correlation terms captured in equations (10) to (14). Intertemporal risk aversion in these terms captures how intertemporal risk aversion *with respect to correlated overall growth* affects the value of the marginal project. In contrast, the ambiguous second

³⁰Note that Weitzman (2009) puts the prior on the variance σ rather than on the expected value of growth. He loosely relates the uncertainty to climate sensitivity. The above is a significantly simplified, but insightful, perspective on Weitzman's approach – abstracting from learning.

order distribution is not directly correlated to the project payoff and, thus, ambiguity aversion only interacts directly with the variance of the expected economic growth.

Current estimates of the parameter RAA in the KMM model are significantly less reliable than in the intertemporal risk aversion framework, and I refrain from a numerical analysis. Moreover, these models do not simultaneously estimate aversion to risk, ambiguity, and intertemporal substitution.³¹ However, the similarity of the ambiguity aversion effect to the direct effect of intertemporal risk aversion gives a good feeling for the magnitude, by which a given degree of relative ambiguity aversion changes the social discount rate. Instead of redoing these simulations for ambiguity aversion, I will explore the effects of ambiguity about correlation between project payoffs and economic growth.

4.4 The Social Discount Rate and Uncertainty about Correlation

In section 3.2, I discussed different arguments in favor of positive, negative, and no correlation between climate change related project payoffs and baseline growth risk. In this subsection, I introduce uncertainty about correlation. Taking the opposite extreme of a perfectly known correlation, I assume an uniform prior over the correlation coefficient, which permits an analytic solution. I am particularly interested in the difference between complete ignorance about the correlation and a known absence of correlation. The first order distribution capturing risk is a joint normal over project productivity and economic growth as in the previous sections. I assume that the correlation κ , between project productivity $\ln y$ and economic growth g , is uniformly distributed on $[-1, 1]$.

Proposition 4: The stochastic social discount rate in the isoelastic setting with intertemporal risk aversion and a uniform prior over correlation is

$$r = \delta + \eta\mu_g - \eta^2 \frac{\sigma_g^2}{2} - \text{RIRA} \left| 1 - \eta^2 \right| \frac{\sigma_g^2}{2} - \ln \left[\frac{\sinh \{ \eta \sigma_g \sigma_y + |1 - \eta| \text{RIRA} \sigma_g \sigma_y \}}{\eta \sigma_g \sigma_y + |1 - \eta| \text{RIRA} \sigma_g \sigma_y} \right]. \quad (15)$$

The terms in the first line resemble the risk-free social discount rate under intertemporal risk aversion derived in section 3. The second line captures the effect of uncertainty about

³¹Paralleling this paper is a work by Ju & Miao (2009) using a similar model of threefold disentanglement. However, the authors fix Arrow-Pratt risk aversion exogenously to a level significantly lower than in the cited estimates of Vissing-Jørgensen & Attanasio (2003), Basal & Yaron (2004), and Basal et al. (2010), and then find an ambiguity measure in the range these papers estimate for standard risk aversion.

the project and its correlation with baseline growth. This additional component is of the form $h(z) = \ln \left[\frac{\sinh\{z\}}{z} \right]$, non-negative, and always reduces the discount rate as long as $z = (\eta + |1 - \eta| \text{RIRA}) \sigma_g \sigma_y \neq 0$. This latter condition, $z \neq 0$, is satisfied as long as the project and the baseline are stochastic and preferences do not simultaneously satisfy $\eta = 0$ and $\text{RIRA} = 0$. The function h can be expanded into $h(z) = \frac{z^2}{6} - \frac{z^4}{180} + \frac{z^6}{2835} + O[z^7]$, where the first term already gives a good approximation for the magnitude relevant for the yearly discount rate. In the setting with yearly iid growth uncertainty in section 3.2, I found that z was below one percent in all scenarios, which makes $h(z)$ negligible. Observe that ambiguity aversion does not affect the social discount rate in equation (15). Only the first order probability distribution (risk) affects overall growth, and only aversion the overall growth uncertainty finds its way into the social discount rate, valuing a marginal project. Ambiguity over the correlation coefficient merely affects the project payoff in second order and has no effect on overall economic growth. Therefore, ambiguity *aversion* has no impact on the social discount rate in the case of an ambiguous correlation parameter.

In the 50 year scenario, ignorance about correlation still only implies a small deviation from the case of no correlation. I adopt a pure time preference of $\delta = 1.5\%$ and the growth scenario introduced in section 3.3. I assume a probability of $p^* = 0.1\%$ that society will be worse off in 50 years than today. Then, in the disentangled scenario D, ignorance over correlation reduces the average discount rate from an uncorrelated 1.3% to 1.2%.³² In the first sensitivity scenario (S1), where $\text{RRA} = 9.5$ and $\eta = 2$, it reduces the average discount rate from 2.7% to 2.6%. The differences in the second sensitivity scenario (average rate of 4.1%) and in Nordhaus's scenario (average rate of 5.1%) are negligible. The difference between the assumptions of ignorance over correlation and no correlation grows as the risk increases. For $p^* = 0.5\%$, ignorance as opposed to being uncorrelated reduces the average rate from 0.6% to 0.4% in the disentangled D scenario, and from 1.5% to 1.3% in the S1 scenario, still leaving the last digit unchanged in the S2 and the N scenarios (with average rates of 3.5% (S2) and 4.9% (N), respectively). Thus, for moderate growth risk, the intertemporal evaluation of uncorrelated stochastic projects and projects with ignorance over the correlation coefficient are both well approximated by the risk-free social discount rate. Then, only a reasonably well founded estimate of a non-trivial correlation will have a major impact on the evaluation. For larger growth risk, the function $h(z)$ captures a small precautionary reduction of the growth rate when correlation is unknown.

³²The numbers represent the risk free rate calculated in equation (8), and the risk free rate less the term $h(\cdot)$ evaluated for $\sigma_g = \sigma_y = 0.3236$ corresponding to $p^* = 0.1\%$.

5 Conclusions

The paper shows a close similarity between the recent smooth ambiguity model and a reformulated version of the well-known Epstein-Zin-Weil model. Both models disentangle uncertainty attitude from the propensity to smooth consumption over time. The smooth ambiguity model does so by capturing an additional intrinsic uncertainty aversion when evaluating subjective second order uncertainty. Analogously, intertemporal risk aversion captures an additional intrinsic aversion to risk in the reformulated Kreps-Porteus-Epstein-Zin-Weil model. I combined the models, obtaining a threefold disentanglement between risk aversion, intertemporal consumption smoothing, and ambiguity aversion. The original version of the intertemporal smooth ambiguity model, in contrast, entangles whether ambiguity aversion captures a difference from risk attitude or a difference from consumption smoothing attitude. I showed how the similarity of the two models and their unification translates into the resulting discount rates.

Long-term investment projects are subject to major uncertainties. The assessment of climate change is an important example. The recent discussion following the Stern review has shown the primordial importance of the discount rate. This discussion is framed almost exclusively in the standard intertemporally additive expected utility setting. I pointed out several omitted contributions, three of which already arise in settings of pure risk. First, decoupling Arrow-Pratt risk aversion from intertemporal substitutability lowers the growth effect in the social discount rate. This increase in future weight is a consequence of the empirical finding that the aversion to intertemporal consumption change is overestimated when the parameter simultaneously has to capture the generally stronger aversion to risk. Second, decoupling risk aversion from the a priori independent preference for consumption smoothing also removes an implicit assumption of intrinsic or intertemporal risk neutrality. I showed that a term proportional to the coefficient of relative intertemporal risk aversion further reduces the risk-free social discount rate. The third contribution is for stochastic projects, where payoffs are correlated to the economic baseline. I showed that, for moderate risks, ignorance about correlation only implies a small precautionary effect reducing the discount rate. However, well understood correlation has a major influence on cost benefit analysis and intertemporal evaluation. I showed that, under general uncertainty attitude, a small intertemporally correlated risk suffices to cut the discount rate back to pure time preference, eliminating the major growth effect in discounting. The application focused on analytic extensions on the Ramsey rule with simple tractable solutions. These formulas invite back of the envelope calculations determining the relative importance of the different discounting

contributions. Accounting for uncertainty in large-scale economic models is usually computationally expensive. The derived formulas can guide a cost benefit analysis of incorporating uncertainty into integrated assessment models.

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Appendix A

Notation: The calculations in the appendix make frequent use of the abbreviations $\rho = 1 - \eta$ and $\alpha = 1 - \text{RRA}$, characterizing the exponents of the isoelastic aggregators.

Proof of Proposition 1: The first part of the proof calculates the marginal value of an additional certain unit of consumption in the second period (dx_2), expressed in terms of marginal first period consumption units (dx_1). This value derives from the marginal trade-off that leaves aggregate welfare

$$U(x_1, p) = \frac{x_1^\rho}{\rho} + \beta \frac{1}{\rho} [E_p x_2^\alpha]^\frac{\rho}{\alpha}$$

unchanged:

$$\begin{aligned} dU(x_1, p) &= x_1^{\rho-1} dx_1 + \beta \frac{1}{\alpha} [E_p x_2^\alpha]^\frac{\rho}{\alpha}-1 E_p \alpha x_2^{\alpha-1} dx_2 \stackrel{!}{=} 0 \\ \Rightarrow x_1^{\rho-1} dx_1 &= -\beta [E_p x_2^\alpha]^\frac{\rho}{\alpha}-1 E_p x_2^{\alpha-1} dx_2 \\ \Rightarrow \frac{dx_1}{dx_2} &= -\beta \left[E_p \left(\frac{x_2}{x_1} \right)^\alpha \right]^\frac{\rho}{\alpha}-1 E_p \left(\frac{x_2}{x_1} \right)^{\alpha-1} \\ \Rightarrow \frac{dx_1}{dx_2} &= -\beta \left[E_p e^{\alpha \ln \frac{x_2}{x_1}} \right]^\frac{\rho}{\alpha}-1 E_p e^{(\alpha-1) \ln \frac{x_2}{x_1}} \\ \Rightarrow \frac{dx_1}{dx_2} &= -\beta \left[e^{\alpha \mu + \alpha^2 \frac{\sigma^2}{2}} \right]^\frac{\rho}{\alpha}-1 e^{(\alpha-1)\mu + (1-\alpha)^2 \frac{\sigma^2}{2}} \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \frac{dx_1}{dx_2} &= -\beta e^{\rho\mu + \alpha\rho\frac{\sigma^2}{2} - \alpha\mu - \alpha^2\frac{\sigma^2}{2}} e^{(\alpha-1)\mu + (1-\alpha)^2\frac{\sigma^2}{2}} \\
 \Rightarrow \frac{dx_1}{dx_2} &= -\beta e^{(\rho-1)\mu + (\alpha\rho+1-2\alpha)\frac{\sigma^2}{2}} \\
 \Rightarrow \frac{dx_1}{dx_2} &= -\beta e^{(\rho-1)\mu + (\alpha\rho+1-2\alpha)\frac{\sigma^2}{2}}.
 \end{aligned}$$

The second part of the proof translates the relation into rates by defining the social discount rate $r = -\ln \frac{dx_1}{-dx_2}$ ($= -\ln \frac{dx_2}{-dx_1}|_{\bar{U}}$), the rate of pure time preference $\delta = -\ln \beta$, and $\eta = 1 - \rho$ ($= \frac{1}{\sigma}$). Further below, I make use of the relation $1 = \frac{1-\eta}{\rho}$.

$$\Rightarrow r = \delta + (1 - \rho)\mu - (\alpha(\rho - 1) + 1 - \alpha)\frac{\sigma^2}{2} \quad (\text{A.1})$$

$$\Rightarrow r = \delta + \eta\mu - \eta^2\frac{\sigma^2}{2} + (\eta^2 + \alpha(\eta + 1) - 1)\frac{\sigma^2}{2}$$

$$\Rightarrow r = \delta + \eta\mu - \eta^2\frac{\sigma^2}{2} + (\eta^2 + \frac{\alpha}{\rho}(1 - \eta)(\eta + 1) - 1)\frac{\sigma^2}{2}$$

$$\Rightarrow r = \delta + \eta\mu - \eta^2\frac{\sigma^2}{2} + (\eta^2 + \frac{\alpha}{\rho}(1 - \eta^2) - 1)\frac{\sigma^2}{2}$$

$$\Rightarrow r = \delta + \eta\mu - \eta^2\frac{\sigma^2}{2} - (1 - \frac{\alpha}{\rho})(1 - \eta^2)\frac{\sigma^2}{2}$$

$$\Rightarrow r = \delta + \eta\mu - \eta^2\frac{\sigma^2}{2} - \text{RIRA} \left| 1 - \eta^2 \right| \frac{\sigma^2}{2}. \quad (\text{A.2})$$

□

Proof of Proposition 2: For the isoelastic specification and with the definition

$$U_2(\epsilon) = f^{-1} [E_{p(x_2, y)} f \circ u(x_2 + \epsilon y)] = \frac{1}{\rho} [E_{p(x_2, y)} (x_2 + \epsilon y)^\alpha]^\frac{\rho}{\alpha}$$

equation (9) translates into

$$x_1^{\rho-1} dx_1 + \beta \frac{d}{d\epsilon} U_2(\epsilon) \Big|_{\epsilon=0} d\epsilon \stackrel{!}{=} 0 \quad (\text{A.3})$$

In order to calculate $\frac{d}{d\epsilon} U_2(\epsilon) \Big|_{\epsilon=0} d\epsilon$ the following definition is useful.

$$V_\epsilon(a, b) = E_{p(x_2, y)} (x_2 + \epsilon y)^a y^b. \quad (\text{A.4})$$

Then

$$\begin{aligned} \left. \frac{d}{d\epsilon} U_2(\epsilon) \right|_{\epsilon=0} &= \frac{1}{\alpha} V_\epsilon(\alpha, 0)^{\frac{\rho}{\alpha}-1} \alpha V_\epsilon(\alpha-1, 1) \Big|_{\epsilon=0} \\ &= V_0(\alpha, 0)^{\frac{\rho}{\alpha}-1} V_0(\alpha-1, 1) \end{aligned} \quad (\text{A.5})$$

where equality between the first and the second line follows from Lebesgue's dominated convergence theorem. Analogously to step 1 in the proof of Proposition 1, I calculate with $z = \ln y$

$$\begin{aligned} V_0(\alpha, 0) &= x_1^\alpha \mathbb{E}_{p(x_2, y)} \left(\frac{x_2}{x_1} \right)^\alpha = x_1^\alpha \mathbb{E}_{p(g, z)} e^{\alpha g} \\ &= x_1^\alpha \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\alpha g} \frac{e^{-\frac{1}{2(1-\kappa^2)} \left[\left(\frac{g-\mu_g}{\sigma_g} \right)^2 + \left(\frac{z-\mu_y}{\sigma_y} \right)^2 - 2\kappa \left(\frac{g-\mu_g}{\sigma_g} \right) \left(\frac{z-\mu_y}{\sigma_y} \right) \right]}}{2\pi\sigma_g\sigma_y\sqrt{1-\rho^2}} dg dz \end{aligned} \quad (\text{A.6})$$

$$= x_1^\alpha e^{\alpha\mu_g + \alpha^2 \frac{\sigma_g^2}{2}}. \quad (\text{A.7})$$

Similarly,

$$\begin{aligned} V_0(\alpha-1, 1) &= x_1^{\alpha-1} \mathbb{E}_{p(x, y)} \left(\frac{x_2}{x_1} \right)^{\alpha-1} y = x_1^{\alpha-1} \mathbb{E}_{p(g, z)} e^{(\alpha-1)g+z} \\ &= x_1^{\alpha-1} e^{-(1-\alpha) \left[\mu_g - (1-\alpha) \frac{\sigma_g^2}{2} + \kappa\sigma_g\sigma_y \right] + \mu_y + \frac{\sigma_y^2}{2}} \end{aligned}$$

so that

$$\begin{aligned} \left. \frac{d}{d\epsilon} U_2(\epsilon) \right|_{\epsilon=0} &= x_1^{\rho-\alpha+\alpha-1} e^{(\alpha\mu_g + \alpha^2 \frac{\sigma_g^2}{2})(\frac{\rho}{\alpha}-1)} e^{-(1-\alpha) \left[\mu_g - (1-\alpha) \frac{\sigma_g^2}{2} + \kappa\sigma_g\sigma_y \right] + \mu_y + \frac{\sigma_y^2}{2}} \\ &= x_1^{\rho-1} e^{(\rho-1)\mu_g + [\alpha(\rho-1) + (1-\alpha)] \frac{\sigma_g^2}{2} - (1-\alpha)\kappa\sigma_g\sigma_y + \mu_y + \frac{\sigma_y^2}{2}}. \end{aligned} \quad (\text{A.8})$$

Substituting the result into equation (A.3) and solving for the discount rate yields

$$\begin{aligned} r = \ln \frac{d\epsilon}{-dx_1} &= \delta + (1-\rho)\mu_g - [\alpha(\rho-1) + 1-\alpha] \frac{\sigma^2}{2} \\ &\quad + (1-\alpha)\kappa\sigma_g\sigma_y - \left(\mu_y + \frac{\sigma_y^2}{2} \right). \end{aligned}$$

The first line corresponds to equation (A.1) and, thus, equation (A.2), yielding the risk-free discount rate under intertemporal risk aversion. Moreover, the random variable y

was assumed to yield an expected value (project payoff) of unity, which implies

$$\mathbb{E}_{p(x,y)} y = e^{\mu_y + \frac{\sigma_y^2}{2}} \stackrel{!}{=} 1 \quad \Rightarrow \quad \mu_y + \frac{\sigma_y^2}{2} = 0,$$

eliminating the last bracket. Finally, $1 - \alpha$ has to be expressed in terms of η (capturing the effects of the standard model) and RIRA (capturing the additional effects of intertemporal risk aversion). I find for $\rho > 0$ that

$$1 - \alpha = 1 - (1 - \eta)(1 - \text{RIRA}) = \eta + (1 - \eta) \text{RIRA}$$

and for $\rho < 0$ that

$$1 - \alpha = 1 - (1 - \eta)(1 + \text{RIRA}) = \eta - (1 - \eta) \text{RIRA}.$$

In both cases this yields

$$1 - \alpha = \eta + |1 - \eta| \text{RIRA}, \tag{A.9}$$

which gives rise to the form stated in the proposition. $\square$

Proof of Corollary 1:

In case 1 of the risk-free discount rate, equation (8) translates $r_{50} = 50\delta$ into the condition $\eta 50\mu \stackrel{!}{=} \eta^2 \frac{\sigma^2}{2} + \text{RIRA} |1 - \eta^2| \frac{\sigma^2}{2}$, which results in the stated equation for σ . Similarly in case 2, equation (10), $\sigma = \sigma_g = \sigma_y$, and $\eta 50\mu_g \stackrel{!}{=} \eta^2 \frac{\sigma_g^2}{2} + \text{RIRA} |1 - \eta^2| \frac{\sigma_g^2}{2} - \eta \kappa \sigma_g \sigma_y - |1 - \eta| \text{RIRA} \kappa \sigma_g \sigma_y$ yield the result. Without the condition $\sigma = \sigma_g = \sigma_y$ the same reasoning gives statement 3 of the corollary. $\square$

Proof of Proposition 3:

Define for the isoelastic specification

$$\begin{aligned} U_2^a(\epsilon) &= \Phi^{-1} \left\{ \int_{\Theta} \Phi \left[f^{-1} \mathbb{E}_{p_{\theta}(x_2, y)} f \circ u(x_2 + \epsilon y) \right] d\pi(\theta) \right\} \\ &= \frac{1}{\rho} \left\{ \int_{\Theta} \left[\mathbb{E}_{p_{\theta}(x_2, y)} (x_2 + \epsilon y)^{\alpha} \right]^{\frac{\rho}{\alpha} \varphi} d\pi(\theta) \right\}^{\frac{1}{\varphi}}. \end{aligned}$$

I have to solve once more the equation

$$dV(x_1, p, \pi) = x_1^{\rho-1} dx_1 + \beta \frac{d}{d\epsilon} U_2^a(\epsilon) \Big|_{\epsilon=0} d\epsilon \stackrel{!}{=} 0 \tag{A.10}$$

for $r = \ln \frac{d\epsilon}{-dx_1}$. Making once more use of the definition

$$V_{\epsilon}(a, b) = \mathbb{E}_{p_{\theta}(x_2, y)} (x_2 + \epsilon y)^a y^b,$$

where θ replaces μ_g in $p_{(x,y)}$ of equations (A.4) and (A.6), I find

$$\begin{aligned}
 \left. \frac{d}{d\epsilon} U_2^a(\epsilon) \right|_{\epsilon=0} &= \frac{1}{\rho} \frac{1}{\varphi} \left\{ \int_{\Theta} V_{\epsilon}(\alpha, 0) \frac{\rho}{\alpha} d\pi(\theta) \right\}^{\frac{1}{\varphi}-1} \\
 &\quad \left\{ \int_{\Theta} \frac{\rho}{\alpha} \varphi V_{\epsilon}(\alpha, 0) \frac{\rho}{\alpha} \alpha V_{\epsilon}(\alpha-1, 1) d\pi(\theta) \right\} \Big|_{\epsilon=0} \\
 &= \left\{ \int_{\Theta} V_0(\alpha, 0) \frac{\rho}{\alpha} d\pi(\theta) \right\}^{\frac{1}{\varphi}-1} \\
 &\quad \int_{\Theta} V_0(\alpha, 0) \frac{\rho}{\alpha} \alpha V_0(\alpha-1, 1) d\pi(\theta). \tag{A.11}
 \end{aligned}$$

With the help of equation (A.7), the expression $\{\cdot\}$ calculates to

$$\begin{aligned}
 \int_{\Theta} x_1^{\alpha(\frac{\rho}{\alpha})} e^{(\alpha\theta + \alpha^2 \frac{\sigma_g^2}{2})(\frac{\rho}{\alpha})} d\pi(\theta) &= x_1^{\rho\varphi} e^{\rho\varphi\alpha \frac{\sigma_g^2}{2}} \int_{\Theta} e^{\rho\varphi\theta} \frac{e^{-\frac{1}{2}(\frac{\theta-\mu_g}{\sigma_g})^2}}{\sqrt{2\pi}\sigma_g} d\theta \\
 &= x_1^{\rho\varphi} e^{\rho\varphi\alpha \frac{\sigma_g^2}{2}} e^{\rho\varphi\mu_g + \rho^2\varphi^2 \frac{\tau_g^2}{2}}.
 \end{aligned}$$

Acknowledging the equality of equations (A.5) and (A.8) and their similarity to the second integrand in equation (A.11) (for $\rho \leftrightarrow \rho\varphi$), this second integral becomes

$$\begin{aligned}
 &\int_{\Theta} V_0(\alpha, 0) \frac{\rho}{\alpha} \alpha V_0(\alpha-1, 1) d\pi(\theta) \\
 &= \int_{\Theta} x_1^{\rho\varphi-1} e^{(\rho\varphi-1)\theta + [\alpha(\rho\varphi-1) + (1-\alpha)] \frac{\sigma_g^2}{2} - (1-\alpha)\kappa\sigma_g\sigma_y + \mu_y + \frac{\sigma_y^2}{2}} d\pi(\theta) \\
 &= x_1^{\rho\varphi-1} e^{[\alpha(\rho\varphi-1) + (1-\alpha)] \frac{\sigma_g^2}{2} - (1-\alpha)\kappa\sigma_g\sigma_y + \mu_y + \frac{\sigma_y^2}{2}} \int_{\Theta} e^{(\rho\varphi-1)\theta} d\pi(\theta) \\
 &= x_1^{\rho\varphi-1} e^{[\alpha(\rho\varphi-1) + (1-\alpha)] \frac{\sigma_g^2}{2} - (1-\alpha)\kappa\sigma_g\sigma_y + \mu_y + \frac{\sigma_y^2}{2}} e^{(\rho\varphi-1)\mu_g + (\rho\varphi-1)^2 \frac{\tau_g^2}{2}}.
 \end{aligned}$$

Substituting these results back into equation (A.11) delivers

$$\begin{aligned}
 \left. \frac{d}{d\epsilon} U_2^a(\epsilon) \right|_{\epsilon=0} &= x_1^{\rho\varphi(\frac{1}{\varphi}-1)} e^{(\rho\varphi\alpha \frac{\sigma_g^2}{2})(\frac{1}{\varphi}-1)} e^{(\rho\varphi\mu_g + \rho^2\varphi^2 \frac{\tau_g^2}{2})(\frac{1}{\varphi}-1)} \\
 &\quad x_1^{\rho\varphi-1} e^{[\alpha(\rho\varphi-1) + (1-\alpha)] \frac{\sigma_g^2}{2} - (1-\alpha)\kappa\sigma_g\sigma_y + \mu_y + \frac{\sigma_y^2}{2}} e^{(\rho\varphi-1)\mu_g + (\rho\varphi-1)^2 \frac{\tau_g^2}{2}} \\
 &= x_1^{\rho-1} e^{[\alpha(\rho-1) + (1-\alpha)] \frac{\sigma_g^2}{2} + (\rho-1)\mu_g + [\rho\varphi(\rho-1) + 1 - \rho\varphi] \frac{\tau_g^2}{2} - (1-\alpha)\kappa\sigma_g\sigma_y + \mu_y + \frac{\sigma_y^2}{2}}.
 \end{aligned}$$

Substituting this result into equation (A.10) and solving for $r = \ln \frac{d\epsilon}{-dx_1}$ yields analogously to the proof of Proposition 2 the discount rate

$$r = \delta + \eta\mu_g - \eta^2 \frac{\sigma_g^2}{2} - \text{RIRA} \left| 1 - \eta^2 \right| \frac{\sigma_g^2}{2} + \eta \kappa \sigma_g \sigma_y + |1 - \eta| \text{RIRA} \kappa \sigma_g \sigma_y - [1 - 2\rho\varphi + \rho^2\varphi] \frac{\tau_g^2}{2}.$$

The last term can be rearranged to the form

$$\begin{aligned} [1 - 2\rho\varphi + \rho^2\varphi] \frac{\tau_g^2}{2} &= [(1 - \varphi) + \varphi(1 - \rho) - \varphi\rho(1 - \rho)] \frac{\tau_g^2}{2} \\ &= [(1 - \varphi) + (1 - \rho)^2 + (\varphi - 1)(1 - \rho)^2] \frac{\tau_g^2}{2} \\ &= [\eta^2 + (1 - \varphi)(1 - \eta^2)] \frac{\tau_g^2}{2} = \eta^2 \frac{\tau_g^2}{2} + \text{RAA} \left| 1 - \eta^2 \right| \frac{\tau_g^2}{2}, \end{aligned}$$

completing the proof. $\square$

Proof of Proposition 4:

Up to equation (A.11) the proof is identical to that of Proposition 3. In the next step, in $V_0(\alpha, 0)^{\frac{\rho\varphi}{\alpha}}$ the ambiguity parameter θ replaces κ instead of μ_g . Thus, the first integral in equation (A.11) becomes

$$\begin{aligned} \int_{\Theta} x_1^{\alpha(\frac{\rho\varphi}{\alpha})} e^{(\alpha\mu_g + \alpha^2 \frac{\sigma_g^2}{2})(\frac{\rho\varphi}{\alpha})} d\pi(\theta) &= x_1^{\rho\varphi} e^{\rho\varphi\mu_g + \rho\varphi\alpha \frac{\sigma_g^2}{2}} \int_{-1}^1 \frac{1}{2} d\theta \\ &= x_1^{\rho\varphi} e^{\rho\varphi\mu_g + \rho\varphi\alpha \frac{\sigma_g^2}{2}}. \end{aligned}$$

For the integrand of the second integral in equation (A.11), I find

$$V_0(\alpha - 1, 1) = x_1^{\alpha-1} e^{(\alpha-1)\mu_g + (\alpha-1)^2 \frac{\sigma_g^2}{2} + (\alpha-1)\theta\sigma_g\sigma_y + \mu_y + \frac{\sigma_y^2}{2}}$$

delivering the integral

$$\begin{aligned}
 & \int_{\Theta} V_0(\alpha, 0)^{\frac{\rho\varphi}{\alpha}-1} V_0(\alpha-1, 1) d\pi(\theta) \\
 &= \int_{\Theta} x_1^{\rho\varphi-\alpha} e^{\rho\varphi\mu_g + \rho\varphi\alpha\frac{\sigma_g^2}{2} - \alpha\mu_g - \alpha^2\frac{\sigma_g^2}{2}} \\
 & \quad x_1^{\alpha-1} e^{(\alpha-1)\mu_g + (\alpha-1)^2\frac{\sigma_g^2}{2} + (\alpha-1)\theta\sigma_g\sigma_y + \mu_y + \frac{\sigma_y^2}{2}} d\pi(\theta) \\
 &= x_1^{\rho\varphi-1} e^{(\rho\varphi-1)\mu_g + (\rho\varphi\alpha-2\alpha-1)\frac{\sigma_g^2}{2} + \mu_y + \frac{\sigma_y^2}{2}} \int_{-1}^1 e^{(\alpha-1)\theta\sigma_g\sigma_y} \frac{1}{2} d\theta \\
 &= x_1^{\rho\varphi-1} e^{(\rho\varphi-1)\mu_g + (\rho\varphi\alpha-2\alpha-1)\frac{\sigma_g^2}{2} + \mu_y + \frac{\sigma_y^2}{2}} \frac{\sinh[(\alpha-1)\sigma_g\sigma_y]}{(\alpha-1)\sigma_g\sigma_y}.
 \end{aligned}$$

Substituting these results back into equation (A.11) returns the second period welfare change in ϵ :

$$\begin{aligned}
 \left. \frac{d}{d\epsilon} U_2^a(\epsilon) \right|_{\epsilon=0} &= x_1^{\rho\varphi(\frac{1}{\varphi}-1)} e^{(\rho\varphi\mu_g + \rho\varphi\alpha\frac{\sigma_g^2}{2})(\frac{1}{\varphi}-1)} \\
 & \quad x_1^{\rho\varphi-1} e^{(\rho\varphi-1)\mu_g + (\rho\varphi\alpha-2\alpha-1)\frac{\sigma_g^2}{2} + \mu_y + \frac{\sigma_y^2}{2}} \frac{\sinh[(\alpha-1)\sigma_g\sigma_y]}{(\alpha-1)\sigma_g\sigma_y} \\
 &= x_1^{\rho-1} e^{(\rho-1)\mu_g + (\rho\alpha-2\alpha-1)\frac{\sigma_g^2}{2} + \mu_y + \frac{\sigma_y^2}{2}} \frac{\sinh[(\alpha-1)\sigma_g\sigma_y]}{(\alpha-1)\sigma_g\sigma_y}.
 \end{aligned}$$

Substituting this result into equation (A.10) and solving for $r = \ln \frac{d\epsilon}{-dx_1}$ yields analogously to the proof of Proposition 2 the discount rate

$$r = \delta + \eta\mu_g - \eta^2 \frac{\sigma_g^2}{2} - \text{RIRA} \left| 1 - \eta^2 \right| \frac{\sigma_g^2}{2} - \ln \left[\frac{\sinh[(\alpha-1)\sigma_g\sigma_y]}{(\alpha-1)\sigma_g\sigma_y} \right].$$

By symmetry of the hyperbolic sine, the sign of $(\alpha-1)$ can be flipped simultaneously in the numerator and the denominator. Using equation (A.9) to substitute for $(1-\alpha)$ then yields the result stated in the proposition. $\square$

Appendix B

The following proposition formalizes how intertemporal risk aversion, defined in the sense of equation (4), translates into the curvature of the function f in a preference representation of the form (3).³³

Proposition 5: Let preferences over $X \times P$ be represented by equation (3) with a continuous function $u : X \rightarrow \mathbb{R}$ and a strictly increasing and continuous function $f : U \rightarrow \mathbb{R}$, where $U = u(X)$ and $\beta = 1$.

- a) The corresponding decision maker is (weakly) intertemporal risk averse [loving], if and only if, the function f is concave [convex].
- b) The corresponding decision maker is intertemporal risk neutral, if and only if, there exist $a, b \in \mathbb{R}$ such that $f(z) = az + b$. An intertemporal risk neutral decision maker maximizes intertemporally additive expected utility (equation 1).

Proof of Proposition 5: a) *Sufficiency* of axiom (4): The premise of axiom (4) translates with $\beta = 1$ into the representation (3) as

$$\begin{aligned} (x^*, x^*) &\sim (x_1, x_2) \\ \Leftrightarrow u(x^*) + u(x^*) &= u(x_1) + u(x_2) \\ \Leftrightarrow u(x^*) &= \frac{1}{2}u(x_1) + \frac{1}{2}u(x_2) \end{aligned} \tag{A.12}$$

Writing the implication of the axiom in terms of representation (3) yields

$$\begin{aligned} (x^*, x^*) &\succ (x^*, \frac{1}{2}x_1 + \frac{1}{2}x_2) \\ \Leftrightarrow u(x^*) &\geq f^{-1} \left(\frac{1}{2}f \circ u(x_1) + \frac{1}{2}f \circ u(x_2) \right). \end{aligned} \tag{A.13}$$

Combining equations (A.12) and (A.13) returns

$$\frac{1}{2}u(x_1) + \frac{1}{2}u(x_2) \geq f^{-1} \left(\frac{1}{2}f \circ u(x_1) + \frac{1}{2}f \circ u(x_2) \right), \tag{A.14}$$

which for an increasing [decreasing] version of f is equivalent to

$$\Leftrightarrow f \left(\frac{1}{2}u(x_1) + \frac{1}{2}u(x_2) \right) > [<] \frac{1}{2}f \circ u(x_1) + \frac{1}{2}f \circ u(x_2).$$

³³Recasting the proposition for a strictly decreasing continuous function $f : U \rightarrow \mathbb{R}$ turns concavity in statement a) into convexity [and convexity into concavity]. Replacing the definition of intertemporal risk aversion by its strict version given in footnote 14 switches concavity to strict concavity in the statement.

Defining $z_i = u(x_i)$, the equation becomes

$$\Leftrightarrow f\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right) \geq [\leq] \frac{1}{2}f(z_1) + \frac{1}{2}f(z_2). \quad (\text{A.15})$$

Because preferences are assumed to be representable in the form (3), there exists a certainty equivalent x^* to all lotteries $\frac{1}{2}x_1 + \frac{1}{2}x_2$ with $x_1, x_2 \in X$. Taking x^* to be the certainty equivalent, the premise and, thus, equation (A.15) have to hold for all $z_1, z_2 \in u(X)$. Therefore, f has to be concave [convex] on $U(x)$ (Hardy, Littlewood & Polya 1964, 75).

Necessity of axiom (4): The necessity is seen to hold by going backward through the proof of sufficiency above. Strict concavity [convexity] of f with f increasing [decreasing] implies that equation (A.15) and, thus, equation (A.14) have to hold for $z_1, z_2 \in u(X)$. The premise corresponding to (A.12) guarantees that equation (A.14) implies equation (A.13) which yields the implication in condition (4). Replacing $\succeq$ by $\preceq$ and $\geq$ by $\leq$ in the proof above implies that the decision maker is intertemporal risk averse, if and only if, f is convex [for an increasing version of f and concave for f decreasing].

b) The decision maker is intertemporal risk neutral, if and only if, f is concave and convex on $u(X)$, which is equivalent to f being linear.³⁴ However, a linear function f cancels out in representation (3) and makes it identical to the intertemporally additive expected utility standard representation (1).

³⁴Alternatively use $\sim$ and $=$ instead of $\succeq$ and $\geq$ in part a) and use Aczél (1966, 46).

Appendix C

Relation of Proposition 1 and Gollier (2002):

Equation (14) in Gollier (2002) is

$$r = \delta + \eta \operatorname{E} \tilde{g} - \operatorname{RRA}(1 + \eta) \frac{\operatorname{Var}(\tilde{g})}{2},$$

where $\delta = \frac{1}{\beta}$, $\tilde{g} = \frac{\tilde{x}_2}{x_1} - 1$, and β denotes Gollier's utility discount factor. The tilde ($\sim$) marks random variables. In contrast, equation (A.1) for the isoelastic normal case translates into

$$r = \delta + \eta\mu - ((1 - \operatorname{RRA})(-\eta) + \operatorname{RRA})\frac{\sigma^2}{2} = \delta + \eta\mu - \operatorname{RRA}(1 + \eta)\frac{\sigma^2}{2} + \eta\frac{\sigma^2}{2}.$$

Thus, Gollier's approximate formula underestimates the social discount rate. Much of the difference can be traced back to the first step of his derivation, where the Arrow-Pratt approximation of the certainty equivalent (or equivalently the certainty equivalent growth rate) loses a term proportional to $\frac{\sigma^2}{2}$ in a setting with isoelastic preferences and a normal growth rate.

In a setting with CARA-utility and a normal distribution of the growth rate Gollier's approximation does much better. I assume that Gollier's growth rate $\tilde{g} = \frac{\tilde{x}_2}{x_1} - 1 \sim N(\mu, \sigma^2)$, which is equivalent to a distribution of second period consumption $\tilde{x}_2 \sim N(x_1(1 + \mu), x_1^2\sigma^2) \equiv N(\mu_x, \sigma_x^2)$. Moreover, the assumption of CARA aggregators translates into $u(x) = -\frac{\exp(-A_u x)}{A_u}$ for utility and $v(x) = -\frac{\exp(-A_v x)}{A_v}$ for Arrow-Pratt risk aversion, adopting Gollier's notation where A_u is absolute aversion to intertemporal substitution and A_v is absolute aversion to risk in the Arrow-Pratt sense. Then, the resulting exact discount rate becomes

$$r = \frac{1}{\beta} \exp \left(A_u \left[\mu_x - x_1 - A_v \frac{\sigma_x^2}{2} \right] \right) \quad (\text{A.16})$$

$$\approx \delta + A_u (\mu_x - x_1) - A_u A_v \frac{\sigma_x^2}{2} = \delta + A_u x_1 \mu - A_u A_v x_1^2 \frac{\sigma^2}{2}. \quad (\text{A.17})$$

The second line is equivalent to Gollier's approximation in his equation (12), before replacing his absolute aversion measures with relative aversion measures. Thus, in the CARA-normal case, the quality of his approximation is given by the quality of the approximation going from equation (A.16) to equation (A.17). For this approximation to hold, risk has to be moderate and expected absolute growth has to be small. Small absolute growth is equivalent to a low expected growth rate or a low present consumption level.